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Chover-type laws of the iterated logarithm for weighted sums

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Abstract

In this paper, a Chover-type law of the iterated logarithm is established for the weighted sums of independent and identically distributed random variables with a distribution in the domain of attraction of a stable law.

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1. Introduction

Let $\{X, X_n, n \geqslant 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with a distribution function $F(x) = P(X \leqslant x)$, and h be a function defined over $[0, 1]$. A weighted sum is defined as

$$S_n(h) = \sum_{k=1}^{n} h\left(\frac{k}{n}\right) X_k.$$

In case $h(x) = 1$ for $x \in [0, 1]$, $S_n := S_n(h) = S_n(1)$ becomes the partial sum.

When h belongs to a certain class of continuous functions over $[0, 1]$, the laws of the iterated logarithm for the weighted sums $S_n(h)$ have been studied by many authors, for example, Gaposkin (1965), Tomkins (1976), Lai and Wei (1982), Stadtmüller (1984), Li and Tomkins (1996). A recent

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result by Li and Tomkins (1996) showed that

$$\limsup_{n\to\infty} \frac{S_n(h)}{(2n\log\log n)^{1/2}} = \left(\int_0^1 h(x)^2\,\mathrm{d}x\right)^{1/2} \quad \text{a.s.} \tag{1.1}$$

if and only if $E(X)=0$ and $E(X^2)=1$. Therefore, if $E(X^2)=\infty$, an analogue of (1.1) is no longer valid. However, for the partial sum S_n, Chover (1966) showed that

$$\limsup_{n\to\infty} \left(\frac{|S_n|}{n^{1/\alpha}}\right)^{1/\log\log n} = \mathrm{e}^{1/\alpha} \quad \text{a.s.} \tag{1.2}$$

when X has a symmetric stable distribution with exponent $\alpha\in(0,2)$, that is,

$$E(\exp(\mathrm{i}tX)) = \exp(-|t|^\alpha) \quad \text{for } t\in R. \tag{1.3}$$

This motivated the study of Chover-type law of the iterated logarithm for weighted sums in Chen (2002). One of the results in Chen (2002) is the following: under condition (1.3) and certain constraints for h,

$$\limsup_{n\to\infty} \left(\frac{|S_n(h)|}{n^{1/\alpha}}\right)^{1/\log\log n} = \mathrm{e}^{1/\alpha} \quad \text{a.s.} \tag{1.4}$$

Note that Qi and Cheng (1996) extended the Chover-type law of the iterated logarithm for the partial sums to the case where the underlying distribution is in the domain of attraction of a stable distribution (see below for details).

Let L_α denote a stable distribution with exponent $\alpha\in(0,2)$. Recall that the distribution of X is said to be in the domain of attraction of L_α if there exist some constants $A_n\in R$ and $B_n>0$ such that

$$\frac{S_n - A_n}{B_n} \xrightarrow{\mathrm{d}} L_\alpha. \tag{1.5}$$

Under (1.5), Qi and Cheng (1996) showed that

$$\limsup_{n\to\infty} \left(\frac{|S_n - A_n|}{B_n}\right)^{1/\log\log n} = \mathrm{e}^{1/\alpha} \quad \text{a.s.}$$

It is easy to verify that in the special case (1.3), (1.5) holds with $A_n=0$ and $B_n=n^{1/\alpha}$, and that L_α has the same distribution as that of X. Based on Chen (2002) and Qi and Cheng (1996), we expect that the Chover-type law of the iterated logarithm for weighted sums holds under the general condition (1.5).

We organize this paper as follows. Our main results are presented in Section 2 and all proofs are postponed till Section 3.

2. Main results

Let $BV[0,1]$ and $B[0,1]$ denote sets of all bounded variation functions and of all bounded functions over $[0,1]$, respectively. As in the introduction, let $\{X, X_n, n\geqslant 1\}$ be a sequence of i.i.d. random variables satisfying (1.5).

It is well known that (1.5) holds if and only if

$$1 - F(x) = \frac{C_1(x)l(x)}{x^\alpha} \quad \text{and} \quad F(-x) = \frac{C_2(x)l(x)}{x^\alpha}, \quad \text{for } x > 0, \tag{2.1}$$

where, for $x > 0$, $C_i(x) \geqslant 0$, $\lim_{x\to\infty} C_i(x) = C_i$, $i = 1, 2$, $C_1 + C_2 > 0$, and $l(x) \geqslant 0$ is a slowly varying function, i.e.,

$$\lim_{t\to\infty} \frac{l(tx)}{l(t)} = 1 \quad \text{for } x > 0.$$

Set $G(x) = P(|X| > x)$, define

$$B(x) = \inf\left\{ y : G(y) \leqslant \frac{1}{x} \right\}, \quad x > 0, \tag{2.2}$$

and write

$$\mu_n(c) = n \int \frac{x}{1 + x^2} \, \mathrm{d}F(cx) \quad \text{for } c > 0.$$

From Loève (1977), (2.1) is equivalent to the statement that $(S_n - \mu_n(B(n)))/B(n)$ converges in distribution to a stable distribution with exponent α. Moreover, we have $\lim_{n\to\infty} B_n/B(n) \in (0, \infty)$ and $\lim_{n\to\infty} (A_n - \mu_n(B(n)))/B(n) \in R$ from convergence of types theorem. In particular, if $\alpha > 1$ then $E(|X|) < \infty$, and one can always choose $B_n = B(n)$ and $A_n = nE(X)$ in (1.5). If $\alpha < 1$ then one can set $A_n = 0$ in (1.5).

Without loss of generalization, we always assume that the condition (1.5) holds with $B(n)$ defined in (2.2) and some constants A_n with $A_0 = 0$. Set $\lg_0(x) = x$ and $\lg_k(x) = \log \max(\lg_{k-1}(x), \mathrm{e})$ for $k \geqslant 1$. Our main theorems are as follows.

Theorem 2.1. *If $h \in BV[0, 1]$ with $h(x_0) \neq 0$ for some $x_0 \in (0, 1]$, and h is continuous at x_0, then under condition (1.5), for any integer $r \geqslant 1$,*

$$\limsup_{n\to\infty} \left(\frac{|S_n(h) - C_n|}{B(\prod_{i=0}^{r-1} \lg_i(n))} \right)^{1/\lg_{r+1}(n)} = \mathrm{e}^{1/\alpha} \quad a.s., \tag{2.3}$$

where

$$C_n = \sum_{k=1}^{n} h\left(\frac{k}{n}\right)(A_k - A_{k-1}).$$

Remark 1. If $\alpha \in (1, 2)$, one can set $A_n = nE(X)$ in Theorem 2.1, and (2.3) holds with

$$C_n = E(X) \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = nE(X) \int_0^1 h(x) \, \mathrm{d}x + \mathrm{O}(1)$$

since $h \in BV[0, 1]$; if $\alpha < 1$, set $A_n = 0$ and then (2.3) holds with $C_n = 0$.

Remark 2. Theorem 2.1 generalizes the result (1.4) in Chen (2002) in two directions: (i) the underlying distributions are allowed from the symmetric stable distributions to those in the domain of attraction of a stable law, and (ii) more accurate results than the law of the iterated logarithm are provided. As a consequence Chover-type law of the iterated logarithm is obtained by taking $r=2$ in Theorem 2.1, i.e.,

$$\limsup_{n\to\infty}\left(\frac{|S_n(h)-C_n|}{B(n)}\right)^{1/\log\log n}=\mathrm{e}^{1/\alpha}\quad \text{a.s.}$$

Next we consider more general weighted sums like Chen (2002) did. More specifically, consider an array of weights $\{a_{n,k}, 1\leqslant k\leqslant n, n\geqslant 1\}$ satisfying conditions

(C1) $\sup_{n\geqslant 1}(\sum_{k=1}^{n-1}|a_{n,k}-a_{n,k+1}|+|a_{n,n}|)<\infty$;

(C2) There exist two increasing sequences of $\{n(k), k\geqslant 1\}$ and $\{m(k), k\geqslant 1\}$ such that $\sup_{k\geqslant 1}(n(k+1)-n(k))<\infty$ and $\liminf_{k\to\infty}|a_{n(k),m(k)}|>0$.

Note that (C1) is satisfied if $\sup_{n\geqslant 1}\max_{1\leqslant k\leqslant n}|a_{n,k}|<\infty$.

Theorem 2.2. *Assume the conditions in Theorem* 2.1 *and conditions* (C1) *and* (C2) *are true. Then*

$$\limsup_{n\to\infty}\left(\frac{|\sum_{k=1}^{n}a_{n,k}X_k-C_n|}{B(\prod_{i=0}^{r-1}\lg_i(n))}\right)^{1/\lg_{r+1}(n)}=\mathrm{e}^{1/\alpha}\quad a.s.$$

for all $r\geqslant 1$, *where*

$$C_n=\sum_{k=1}^{n}a_{n,k}(A_k-A_{k-1}).$$

A straightforward application of Theorem 2.2 with $a_{n,k}=\sum_{i=k}^{n}h(i/n)$ leads to the following theorem.

Theorem 2.3. *Let* $h\in B[0,1]$ *with* $\int_{x_0}^{1}h(x)\neq 0$ *for some* $x_0\in(0,1)$. *Then under condition* (1.5)

$$\limsup_{n\to\infty}\left(\frac{|\sum_{k=1}^{n}h(\frac{k}{n})S_k-C_n|}{B(\prod_{i=0}^{r-1}\lg_i(n))}\right)^{1/\lg_{r+1}(n)}=\mathrm{e}^{1/\alpha}\quad a.s.,$$

for any $r\geqslant 1$, *where*

$$C_n=\sum_{k=1}^{n}h\left(\frac{k}{n}\right)A_k.$$

As another application of Theorem 2.2, we get the following Chover-type law of the iterated logarithm for moving sums.

Theorem 2.4. *Let* $\{b_n\}$ *be a sequence of real numbers satisfying* $0<\sum_{n=1}^{\infty}|b_n|<\infty$. *For* $n\geqslant 1$ *set* $Y_n=\sum_{k=1}^{n}b_{n-k+1}X_k$. *Then under condition* (1.5)

$$\limsup_{n\to\infty}\left(\frac{|\sum_{k=1}^{n}Y_k-C_n|}{B(\prod_{i=0}^{r-1}\lg_i(n))}\right)^{1/\lg_{r+1}(n)}=\mathrm{e}^{1/\alpha}\quad a.s.,$$

for any $r \geqslant 1$, *where*

$$C_n = \sum_{k=1}^{n} b_k(A_k - A_{k-1}).$$

3. Proofs

We need some lemmas before we proceed to the proofs of our theorems.

Lemma 3.1. *Suppose* $l(x)$ *is a slowly varying function at infinity and* $g(x) \geqslant 1$ *is an arbitrary function. Then for any given* $\delta > 0$, *there exists an* $x_0 > 0$ *such that*

$$\frac{1}{2}g^{-\delta}(x) < \inf_{x \leqslant y \leqslant xg(x)} \frac{l(y)}{l(x)} \leqslant \sup_{x \leqslant y \leqslant xg(x)} \frac{l(y)}{l(x)} < 2g^{\delta}(x) \quad \text{for all } x > x_0.$$

Lemma 3.2. *Assume that* $\{Y_n, n \geqslant 1\}$ *is a sequence of i.i.d. random variables and* $\{a_n\}$ *is a sequence of positive constants satisfying*

(a) $\{a_n/n^{1/\gamma}\}$ *is nondecreasing ultimately for some* $\gamma \in (0,2)$;

(b) $\sup_n (a_{2n}/a_n) < \infty$.

If $\sum_n P(|Y_1| > a_n) < \infty$, *then*

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} Y_j - nEY_1 I(|Y_1| \leqslant a_n)}{a_n} = 0 \quad a.s.$$

If $\sum_n P(|Y_1| > a_n) = \infty$, *then*

$$\limsup_{n\to\infty} \frac{|\sum_{j=1}^{n} Y_j - c_n|}{a_n} = \infty \quad a.s.$$

for every sequence $\{c_n\}$.

Lemma 3.1 follows from Bingham et al. (1987, Theorem 1.5.6, p. 25), and Lemma 3.2 follows from Mori (1977, Theorem 1 and Lemmas 1 and 2).

The following lemma plays a central role in the proofs of our theorems.

Lemma 3.3. *Let* $\{p_n\}$ *be a sequence of non-decreasing numbers with* $p_n \geqslant 1$ *and* $\sup_n (p_{2n}/p_n) < \infty$. *Then, under* (1.5), *we have with probability one*

$$\limsup_{n\to\infty} \frac{|S_n - A_n|}{B(np_n)} = 0 \tag{3.1}$$

provided that $\sum_{n=1}^{\infty} (1/np_n) < \infty$.

Proof. Note that under (2.1), $G(x) = P(|X| > x)$ is a regularly varying function with index $-\alpha$ at infinity. From De Haan (1970) or Bingham et al. (1987), $B(x)$ is a regularly varying function with index $1/\alpha$ at infinity, and thus $B(x)$ has the following representation:

$$B(x) = c_1(x)x^{1/a}\exp\left\{\int_1^x \frac{b_1(u)}{u}\,du\right\},$$

where $\lim_{x\to\infty} c_1(x) = c_1 \in (0,\infty)$ and $\lim_{x\to\infty} b_1(x) = 0$. Using the properties of regular variation (see Bingham et al., 1987), we have

$$\lim_{x\to\infty} xG(B(x)) = 1. \tag{3.2}$$

Define

$$g(x) = c_1 x^{1/\alpha}\exp\left\{\int_1^x \frac{b_1(u)}{u}\,du\right\}.$$

It is easy to check that $G(x) \sim g(x)$ as $x \to \infty$ and that for any given $\gamma \in (\alpha, 2)$, there exists an $x_\gamma > 0$ such that $g(x)/x^{1/\gamma}$ is increasing in (x_γ, ∞). Therefore $\lim_{n\to\infty}(B(n)/g(n)) = 1$ and the sequence $a_n = g(np_n)$ satisfies the conditions of Lemma 3.2. Moreover,

$$G(a_n) = G(g(np_n)) \sim G(B(np_n)) \sim \frac{1}{np_n}.$$

This implies that $\sum_{n=1}^\infty P(|X| > a_n) < \infty$. So, from Lemma 3.2,

$$\limsup_{n\to\infty} \frac{|S_n - nE(XI(|X| \leqslant a_n)))|}{B(np_n)} = \limsup_{n\to\infty} \frac{|S_n - nE(XI(|X| \leqslant a_n)|}{a_n} = 0 \quad \text{a.s.} \tag{3.3}$$

Since $p_n \to \infty$ and $B(n) = o(B(np_n))$, we have from (1.5) that

$$\frac{S_n - A_n}{B(np_n)} \stackrel{p}{\to} 0,$$

which, together with (3.3), yields

$$\frac{A_n - nE(XI(|X| \leqslant a_n))}{B(np_n)} \to 0.$$

Hence, the lemma follows from this approximation and (3.3). □

Lemma 3.4. *Let $\{p_n\}$ be a sequence of non-decreasing numbers with $p_n \geqslant 1$ and $\sup_n(p_{2n}/p_n) < \infty$. Suppose $h \in BV[0,1]$ with $h(x_0) \neq 0$ for some $x_0 \in (0,1]$, and h is continuous at x_0. Then under (1.5), with probability one*

$$\limsup_{n\to\infty} \frac{|S_n(h) - C_n|}{B(np_n)} = 0 \tag{3.4}$$

if $\sum_{n=1}^\infty (np_n)^{-1} < \infty$, and

$$\limsup_{n\to\infty} \frac{|S_n(h) - C_n|}{B(np_n)} = \infty \tag{3.5}$$

if $\sum_{n=1}^\infty (np_n)^{-1} = \infty$.

Proof. For any $h \in BV[0,1]$, there exists a constant C such that $|h(x)| \leqslant C$ and $\sum_{k=1}^{n-1} |h(\frac{k}{n}) - h(k+1/n)| \leqslant C$ for all $n \geqslant 1$. Therefore,

$$
\begin{aligned}
|S_n(h) - C_n| &= \left| \sum_{k=1}^{n} h\left(\frac{k}{n}\right)[X_k - (A_k - A_{k-1})] \right| \\
&= \left| \sum_{k=1}^{n-1} \left[h\left(\frac{k}{n}\right) - h\left(\frac{k+1}{n}\right) \right] (S_k - A_k) + h(1)(S_n - A_n) \right| \\
&\leqslant 2C \max_{1 \leqslant k \leqslant n} |S_k - A_k|.
\end{aligned}
$$

Since (3.1) implies $\limsup_{n\to\infty} [(\max_{1\leqslant k\leqslant n} |S_k - A_k|)/B(np_n)] = 0$ due to the fact that $B(np_n)$ is a non-decreasing function of n, (3.4) follows from Lemma 3.3 if $\sum_{n=1}^{\infty} (np_n)^{-1} < \infty$.

Next, we assume $\sum_{n=1}^{\infty} (np_n)^{-1} = \infty$ and shall prove (3.5). It is easily seen that $\limsup_{n\to\infty} [|S_n(h) - C_n|/B(np_n)]$ is measurable with respect to the tail-σ field generated by the sequence $\{X_n, n \geqslant 1\}$ (see Chow and Teicher, 1997, p. 64). From Kolmogorov zero-one law, there is a non-random constant $d \in [0,\infty]$ such that with probability one

$$
\limsup_{n\to\infty} \frac{|S_n(h) - C_n|}{B(np_n)} = d.
$$

So we need to show $d = \infty$. Suppose at the moment that $d < \infty$. Then we will obtain some contradiction.

Since $\sum_{n=1}^{\infty} (np_n)^{-1} = \infty$, there exists a sequence of non-decreasing positive numbers, $\{r_n\}$, such that $r_n \to \infty$ and

$$
\sum_{n=1}^{\infty} \frac{1}{np_n r_n} = \infty. \tag{3.6}
$$

Write $B(x) = x^{1/\alpha} L(x)$, where $L(x)$ is a slowly varying function. By applying Lemma 3.1 with $0 < \delta < 1/\alpha$ we obtain that $B(np_n r_n)/B(np_n) \to \infty$ as $n \to \infty$. Then with probability one,

$$
\lim_{n\to\infty} \frac{|S_n(h) - C_n|}{B(np_n r_n)} = 0.
$$

Let $\{X', X'_n, n \geqslant 1\}$ be an independent copy of $\{X, X_n, n \geqslant 1\}$ and set $X^s = X - X'$ and $X^s_n = X_n - X'_n$ for $n \geqslant 1$. Then

$$
\lim_{n\to\infty} \frac{|\sum_{k=1}^{n} h(\frac{k}{n}) X'_k - C_n|}{B(np_n r_n)} = 0.
$$

Further we have

$$
\lim_{n\to\infty} \frac{|\sum_{k=1}^{n} h(\frac{k}{n}) X^s_k|}{B(np_n r_n)} = 0. \tag{3.7}
$$

Since the random variables X_n^s are symmetric, we could show that (3.7) implies

$$\sum_{n=1}^{\infty} P(|X^s| > B(n p_n r_n)) < \infty \tag{3.8}$$

in a way similar to the proof of Theorem 2.1 of Chen (2002). Further, by using the following inequality:

$$P(|X - m(X)| > x) \leqslant 2P(|X^s| > x) \quad \text{for all } x > 0$$

where $m(X)$ denotes the median of the random variable X (see Chow and Teicher, 1997, Lemma 10.1.1, p. 355), we obtain

$$\sum_{n=1}^{\infty} P(|X - m(X)| > B(n p_n r_n)) < \infty. \tag{3.9}$$

Since

$$G(x + |m(X)|) \leqslant P(|X - m(X)| > x) \leqslant G(x - |m(X)|)$$

for any $x > |m(X)|$, and $G(x)$ is regularly varying, it follows from (3.2) that as $n \to \infty$

$$P(|X - m(X)| > B(n p_n r_n)) \sim G(B(n p_n r_n)) \sim \frac{1}{n p_n r_n}.$$

So (3.9) implies $\sum_{n=1}^{\infty} (n p_n r_n)^{-1} < \infty$, which is in contradiction with (3.6), i.e., $d = \infty$. This completes the proof of this lemma. □

Proof of Theorem 2.1. Let "i.o." denote "infinitely often". It suffices to show that for every $\varepsilon \in (0, 1)$

$$P\left(\left(\frac{|S_n(h) - C_n|}{B(\prod_{i=0}^{r-1} \lg_i(n))}\right)^{1/\lg_{r+1}(n)} > \mathrm{e}^{(1+\varepsilon)/\alpha} \text{ i.o.}\right) = 0$$

and

$$P\left(\left(\frac{|S_n(h) - C_n|}{B(\prod_{i=0}^{r-1} \lg_i(n))}\right)^{1/\lg_{r+1}(n)} > \mathrm{e}^{(1-\varepsilon)/\alpha} \text{ i.o.}\right) = 1,$$

or equivalently

$$P\left(|S_n(h) - C_n| > B\left(\prod_{i=0}^{r-1} \lg_i(n)\right) \mathrm{e}^{((1+\varepsilon)/\alpha)\lg_{r+1}(n)} \text{ i.o.}\right) = 0 \tag{3.10}$$

and

$$P\left(|S_n(h) - C_n| > B\left(\prod_{i=0}^{r-1} \lg_i(n)\right) \mathrm{e}^{((1-\varepsilon)/\alpha)\lg_{r+1}(n)} \text{ i.o.}\right) = 1. \tag{3.11}$$

Take $p_n=[\lg_r(n)]^{\pm\varepsilon_1}\prod_{i=1}^{r}\lg_i(n)$ for $\varepsilon_1=\varepsilon/2$, that is, $np_n=[\lg_r(n)]^{1\pm\varepsilon_1}\prod_{i=0}^{r-1}\lg_i(n)$. Since

$$\sum_{n=1}^{\infty}\frac{1}{[\lg_r(n)]^{1+\varepsilon_1}\prod_{i=0}^{r-1}\lg_i(n)}<\infty$$

and

$$\sum_{n=1}^{\infty}\frac{1}{[\lg_r(n)]^{1-\varepsilon_1}\prod_{i=0}^{r-1}\lg_i(n)}=\infty,$$

it is readily seen from Lemma 3.4 that

$$P\left(|S_n(h)-C_n|>B\left([\lg_r(n)]^{1+\varepsilon_1}\prod_{i=0}^{r-1}\lg_i(n)\right)\ \text{i.o.}\right)=0$$

and

$$P\left(|S_n(h)-C_n|>B\left([\lg_r(n)]^{1-\varepsilon_1}\prod_{i=0}^{r-1}\lg_i(n)\right)\ \text{i.o.}\right)=1.$$

Therefore, we only need to prove that for all large n

$$B\left([\lg_r(n)]^{1+\varepsilon_1}\prod_{i=0}^{r-1}\lg_i(n)\right)<B\left(\prod_{i=0}^{r-1}\lg_i(n)\right)\mathrm{e}^{((1+\varepsilon)/\alpha)\lg_{r+1}(n)} \tag{3.12}$$

and

$$B\left([\lg_r(n)]^{1-\varepsilon_1}\prod_{i=0}^{r-1}\lg_i(n)\right)>B\left(\prod_{i=0}^{r-1}\lg_i(n)\right)\mathrm{e}^{((1-\varepsilon)/\alpha)\lg_{r+1}(n)}. \tag{3.13}$$

As in the proof of Lemma 3.4 write $B(x)=x^{1/\alpha}L(x)$, where $L(x)$ is a slowly varying function. By applying Lemma 3.1 to function $L(x)$ with $d=\varepsilon_1/4$ we get for large n

$$\begin{aligned}\frac{B([\lg_r(n)]^{1+\varepsilon_1}\prod_{i=0}^{r-1}\lg_i(n))}{B(\prod_{i=0}^{r-1}\lg_i(n))}&=[\lg_r(n)]^{(1+\varepsilon_1)/\alpha}\frac{L([\lg_r(n)]^{1+\varepsilon_1}\prod_{i=0}^{r-1}\lg_i(n))}{L(\prod_{i=0}^{r-1}\lg_i(n))}\\&\leqslant 2[\lg_r(n)]^{(1+\varepsilon_1)/\alpha}[\lg_r(n)]^{d(1+\varepsilon_1)/\alpha}\\&\leqslant 2[\lg_r(n)]^{(1+1.5\varepsilon_1)/\alpha}\\&<[\lg_r(n)]^{(1+2\varepsilon_1)/\alpha}\\&=\mathrm{e}^{((1+\varepsilon)/\alpha)\lg_{r+1}(n)},\end{aligned}$$

i.e., (3.12). Likewise, we can show (3.13). This completes the proof. □

Proof of Theorem 2.2. The proof follows from the lines of that of Theorem 2.1. The key is to show Lemma 3.4 holds for weighted sum $\sum_{k=1}^{n} a_{n,k}X_k$. In particular, if condition (C1) holds, then

$$\left|\sum_{k=1}^{n} a_{n,k}X_k - C_n\right| \leqslant C \max_{1\leqslant k\leqslant n} |S_k - A_k|$$

for some constant $C > 0$ (see the proof of Lemma 3.4), and thus (3.4) holds for $\sum_{k=1}^{n} a_{n,k}X_k$; if condition (C2) holds, then $\lim_{n\to\infty} [|\sum_{k=1}^{n} X_k^s|/B(np_nr_n)] = 0$, an analogue of (3.7), yields the equation (3.8) which leads to some contradiction as in the proof of Lemma 3.4, that is, condition (C2) guarantees (3.5). The detailed proof is omitted. □

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