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ARITHMETIC PROPERTIES OF 3-REGULAR PARTITIONS IN THREE COLORS

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Abstract

In 2019, Gireesh and Naika proved an infinite family of congruences modulo powers of 3 for the function $p_{\{3,3\}}(n)$, the number of 3-regular partitions in three colors. In this paper, using elementary generating function manipulations and classical techniques, we significantly extend the list of proven arithmetic properties satisfied by $p_{\{3,3\}}(n)$.

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1. Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers, called parts, whose sum equals n. For ℓ a positive integer, a partition of n is called ℓ-regular if there is no part divisible by ℓ. The generating function for the number of ℓ-regular partitions of n, denoted by $b_\ell(n)$, is given by

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{(q^\ell;q^\ell)_\infty}{(q;q)_\infty},$$

where we use the standard q-series notation (for $|q| < 1$):

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

Arithmetic properties of ℓ-regular partition functions have been studied by many authors, including [3, 5–7, 11–14].

In 2018, Hirschhorn [10] studied the number of partitions of n in three colors, $p_3(n)$, given by

$$\sum_{n=0}^{\infty} p_3(n)q^n = \frac{1}{(q;q)_\infty^3}.$$

He derived a number of congruences for $p_3(n)$ modulo high powers of 3.

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Soon after, Gireesh and Naika [8] studied $p_{\{3,3\}}(n)$, the number of 3-regular partitions in three colors, whose generating function is given by

$$\sum_{n=0}^{\infty} p_{\{3,3\}}(n)q^n = \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty^3}. \tag{1.1}$$

They deduced some congruences modulo powers of 3 for $p_{\{3,3\}}(n)$, including the following: For all $\alpha \geq 0$ and $n \geq 0$,

$$p_{\{3,3\}}\left(3^{2\alpha+1}n + \frac{3^{2\alpha+2}-1}{4}\right) \equiv 0 \pmod{3^{2\alpha+2}}.$$

In this paper, our goal is to significantly extend the list of proven arithmetic properties satisfied by $p_{\{3,3\}}(n)$ using elementary generating function manipulations and well-known q-series identities. In particular, we provide a parity characterization for $p_{\{3,3\}}(2n)$ as well as the following characterization mod 3 for $p_{\{3,3\}}(n)$: For all $n \geq 0$,

$$\begin{aligned} p_{\{3,3\}}(3n+1) &\equiv 0 \pmod{3}, \\ p_{\{3,3\}}(3n+2) &\equiv 0 \pmod{3}, \\ p_{\{3,3\}}(3n) &\equiv \begin{cases} (-1)^{k+l} \pmod{3}, & \text{if } n = k(3k-1)/2 + l(3l-1)/2, \\ 0 \pmod{3}, & \text{otherwise.} \end{cases} \end{aligned}$$

2. Parity characterization for $p_{\{3,3\}}(2n)$

This section is devoted to proving a characterization modulo 2 for $p_{\{3,3\}}(2n)$ as well as some consequences. In order to do so, we need a number of identities.

Throughout this paper, we define

$$f_k := (q^k;q^k)_\infty$$

in order to shorten the notation. Thus, (1.1) becomes

$$\sum_{n=0}^{\infty} p_{\{3,3\}}(n)q^n = \frac{f_3^3}{f_1^3}. \tag{2.1}$$

Lemma 2.1. *The following 2-dissection identities hold:*

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}, \tag{2.2}$$

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}. \tag{2.3}$$

Proof. Identities (2.2) and (2.3) are equations (30.10.3) and (30.9.9) of [9], respectively. ∎

We also recall Jacobi's identity (see [2, Theorem 1.3.9]):

$$f_1^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}. \tag{2.4}$$

Substituting (2.2) and (2.3) into (2.1), we can extract the terms involving q^2 to obtain

$$\sum_{n=0}^{\infty} p_{\{3,3\}}(2n)q^{2n} \equiv \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} \cdot \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} \equiv \frac{f_4^5}{f_2^7} \equiv f_2^3 \pmod{2}.$$

Therefore, thanks to (2.4),

$$\sum_{n=0}^{\infty} p_{\{3,3\}}(2n)q^{n} \equiv f_1^3 \equiv \sum_{k=0}^{\infty} q^{k(k+1)/2} \pmod{2}.$$

Thus, we know

Theorem 2.2. *For all $n \geq 1$,*

$$p_{\{3,3\}}(2n) = \begin{cases} 1, & \text{if } n = k(k+1)/2 \text{ for some } k \in \mathbb{Z}; \\ 0, & \text{otherwise.} \end{cases}$$

We close this section with two consequences of the theorem above.

Corollary 2.3. *Let $p \geq 5$ be a prime and $1 \leq r \leq p-1$ be an integer such that $8r+1$ is a quadratic nonresidue modulo p. Then, for all $n \geq 0$,*

$$p_{\{3,3\}}(2(pn+r)) \equiv 0 \pmod{2}.$$

Proof. We need to know whether $pn + r = k(k+1)/2$, for some $k \in \mathbb{Z}$, which is equivalent to $8(pn+r)+1 = (2k+1)^2$. This implies that $8r+1$ is a quadratic residue modulo p, which contradicts the fact that $8r+1$ is a quadratic nonresidue modulo p. □

Corollary 2.4. *For all $n \geq 0$, $p_{\{3,3\}}(2(3n+2)) \equiv 0 \pmod{2}$.*

Proof. If $8(3n+2)+1 = (2k+1)^2$, for some $k \in \mathbb{Z}$, then $24n+17 = (2k+1)^2$, which would imply that $(2k+1)^2 \equiv 5 \pmod{12}$. However, no square can be congruent to 5 (mod 12). □

3. Congruences modulo powers of 3

With the goal of extending the work of Gireesh and Naika [8] in a slightly different direction, we begin this section by providing a complete characterization for $p_{\{3,3\}}(n)$ modulo 3.

Theorem 3.1. *For all $n \geq 0$,*

$$p_{\{3,3\}}(3n+1) \equiv 0 \pmod{3},$$
$$p_{\{3,3\}}(3n+2) \equiv 0 \pmod{3},$$
$$p_{\{3,3\}}(3n) \equiv \begin{cases} (-1)^{k+l} \pmod{3}, & \text{if } n = k(3k-1)/2 + l(3l-1)/2, \\ 0 \pmod{3}, & \text{otherwise}. \end{cases}$$

Proof. From (2.1) we have

$$\sum_{n=0}^{\infty} p_{\{3,3\}}(n) q^n \equiv f_3^2 \pmod{3}. \tag{3.1}$$

Thus, the coefficients of the terms of the forms q^{3n+1} and q^{3n+2} on both sides of (3.1) are congruent to 0 modulo 3. This proves the first two congruences above.

Extracting the terms of the form q^{3n} from (3.1), we obtain

$$\sum_{n=0}^{\infty} p_{\{3,3\}}(3n) q^{3n} \equiv f_3^2 \pmod{3}.$$

Replacing q^3 by q, it follows that

$$\sum_{n=0}^{\infty} p_{\{3,3\}}(3n) q^n \equiv f_1^2 = \sum_{k,l=-\infty}^{\infty} (-1)^{k+l} q^{k(3k-1)/2 + l(3l-1)/2} \pmod{3}, \tag{3.2}$$

thanks to Euler's identity [9, Eq. (1.6.1)]

$$f_1 = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}. \tag{3.3}$$

Comparing the coefficients of q^n on both sides of (3.2) completes the proof. □

The proof of the next theorem requires the following lemma, which can easily be proved by the binomial theorem.

Lemma 3.2. *Given a prime p, we have*

$$f_1^{p^2} \equiv f_p^p \pmod{p^2}.$$

The next theorem presents an infinite family of congruences modulo 9.

Theorem 3.3. *Let p be a prime such that $p \equiv 3 \pmod{4}$. Then, for all $k, m \geq 0$ with $p \nmid m$, we have*

$$p_{\{3,3\}}\left(p^{2k+1} m + \frac{p^{2k+2}-1}{4}\right) \equiv 0 \pmod{9}.$$

Proof. From (2.1), (2.4), and Lemma 3.2 we see that

$$\sum_{n=0}^{\infty} p_{\{3,3\}}(n)q^n \equiv f_1^6 = (f_1^3)^2 = \sum_{k,l=0}^{\infty}(-1)^{k+l}(2k+1)(2l+1)q^{k(k+1)/2+l(l+1)/2} \pmod 9.$$

Thus $p_{\{3,3\}}(n) \equiv 0 \pmod 9$ if $8n+2$ is not a sum of two squares. However we have $n = p^{2k+1}m + \frac{p^{2k+2}-1}{4}$, which yields

$$8n+2 = 8p^{2k+1}m + 2p^{2k+2} = 2p^{2k+1}(4m+p).$$

We recall that a positive integer N is the sum of two squares if and only if each prime factor congruent to 3 modulo 4 has an even power in the prime factorization of N. Thus, since $p \equiv 3 \pmod 4$, it follows that $8n+2$ is not a sum of two squares, which completes the proof. □

For example, the following congruences are special cases of Theorem 3.3:

$$\begin{aligned} p_{\{3,3\}}(9n+3r+2) &\equiv 0 \pmod 9, \text{ for } r \in \{1,2\},\\ p_{\{3,3\}}(49n+7r+12) &\equiv 0 \pmod 9, \text{ for } r \in \{1,2,\ldots,6\},\\ p_{\{3,3\}}(121n+11r+30) &\equiv 0 \pmod 9, \text{ for } r \in \{1,2,\ldots,10\}. \end{aligned}$$

The rest of this section is devoted to proving an infinite family of congruences modulo 81 for $p_{\{3,3\}}(n)$. We begin by recalling Ramanujan's theta functions

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \text{ for } |ab|<1,$$

$$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{f_2^5}{f_1^2 f_4^2}, \text{ and} \tag{3.4}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{f_2^2}{f_1}. \tag{3.5}$$

We also recall Identity (14) of [4]:

$$\frac{1}{f_1^3} = \frac{f_9^3}{f_3^{12}}\left(P(q^3)^2 + 3qf_9^3P(q^3) + 9q^2f_9^6\right), \tag{3.6}$$

where

$$P(q) = f_1\left(\frac{\varphi(-q^3)^3}{\varphi(-q)} + 4q\frac{\psi(q^3)^3}{\psi(q)}\right).$$

Theorem 3.4. *Let p be a prime such that $p \equiv 3 \pmod 4$. Then, for all $k,m \geq 0$ with $p \nmid m$, we have*

$$p_{\{3,3\}}\left(9p^{2k+1}m + 9\frac{(p^{2k+2}-1)}{4} + 2\right) \equiv 0 \pmod{81}.$$

Proof. Thanks to (3.6) we can extract the terms involving q^{3n+2} from (2.1), which yields

$$\sum_{n=0}^{\infty} p_{\{3,3\}}(3n+2)q^{3n+2} = 9q^2\frac{f_9^9}{f_3^9}.$$

After dividing both sides of the identity above by q^2, replacing q^3 by q, and using Lemma 3.2, we are left with

$$\sum_{n=0}^{\infty} p_{\{3,3\}}(3n+2)q^n \equiv 9f_3^6 \pmod{81}. \tag{3.7}$$

It follows that

$$\sum_{n=0}^{\infty} p_{\{3,3\}}(9n+2)q^n \equiv 9(f_1^3)^2 \pmod{81}.$$

By (2.4), we see that

$$\sum_{n=0}^{\infty} p_{\{3,3\}}(9n+2)q^n \equiv 9\sum_{k,l=0}^{\infty}(-1)^{k+l}(2k+1)(2l+1)q^{k(k+1)/2+l(l+1)/2} \pmod{81}.$$

Note that $n = k(k+1)/2 + l(l+1)/2$ is equivalent to $8n+2 = (2k+1)^2 + (2l+1)^2$. Thus $p_{\{3,3\}}(9n+2) \equiv 0 \pmod{81}$ if $8n+2$ is not a sum of two squares. However we have $n = p^{2k+1}m + \dfrac{p^{2k+2}-1}{4}$, which yields

$$8n+2 = 8p^{2k+1}m + 2p^{2k+2} = 2p^{2k+1}(4m+p).$$

Therefore, $8n+2$ is not a sum of two squares, which completes the proof. □

For example, the following congruences are special cases of Theorem 3.4:

$$\begin{aligned} p_{\{3,3\}}(729n+243r+182) &\equiv 0 \pmod{81}, \text{ for } r \in \{1,2\},\\ p_{\{3,3\}}(441n+63r+110) &\equiv 0 \pmod{81}, \text{ for } r \in \{1,2,\dots,6\},\\ p_{\{3,3\}}(1089n+99r+272) &\equiv 0 \pmod{81}, \text{ for } r \in \{1,2,\dots,10\}. \end{aligned}$$

Corollary 3.5. *For all $n \geq 0$, $p_{\{3,3\}}(9n+5) \equiv p_{\{3,3\}}(9n+8) \equiv 0 \pmod{81}$.*

Proof. These congruences follow from (3.7) after extracting the terms involving q^{3n+1} and q^{3n+2}. □

4. Congruences modulo 4

In order to prove the main result of this section, we need the following identity.

Lemma 4.1.

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8} \tag{4.1}$$

PROOF. By Entry 25 (i), (ii), (v), and (vi) in [1, p. 40], we have

$$\phi(q) = \phi(q^4) + 2q\psi(q^8), \tag{4.2}$$
$$\phi(q)^2 = \phi(q^2)^2 + 4q\psi(q^4)^2. \tag{4.3}$$

Using (3.4) and (3.5) we can rewrite (4.2) in the form

$$\frac{f_2^5}{f_1^2 f_4^2} = \frac{f_8^5}{f_4^2 f_{16}^2} + 2q\frac{f_{16}^2}{f_8},$$

from which we obtain (4.1) after multiplying both sides by $\frac{f_4^2}{f_2^5}$. □

We now prove a small set of congruences modulo 4 which are satisfied by $p_{\{3,3\}}(n)$ for specific arithmetic progressions.

THEOREM 4.2. *For all $n \geq 0$ and $t \in \{16, 46, 76, 136\}$, we have*

$$p_{\{3,3\}}(150n + t) \equiv 0 \pmod{4}.$$

PROOF. Thanks to (3.6) we can extract the terms involving q^{3n+1} from (2.1), which yields

$$\sum_{n=0}^{\infty} p_{\{3,3\}}(3n+1)q^{3n+1} \equiv 3q\frac{f_9^6}{f_3^8}\frac{\phi(-q^9)^3}{\phi(-q^3)} \pmod{4}.$$

After dividing both sides of the congruence above by q, replacing q^3 by q, and using the elementary facts $f_k^4 \equiv f_{2k}^2 \pmod{4}$ and $2f_k^2 \equiv 2f_{2k} \pmod{4}$, we are left with

$$\sum_{n=0}^{\infty} p_{\{3,3\}}(3n+1)q^n \equiv 3\frac{f_3^6}{f_1^8}\frac{\phi(-q^3)^3}{\phi(-q)} \equiv 3\frac{f_2 f_3^{12}}{f_1^{10} f_6^3} \equiv 3\frac{f_2 f_6^3}{f_1^{10}} \equiv 3\frac{f_6^3}{f_1^2 f_2^3} \pmod{4}.$$

Now we use (4.1) to extract the odd part on both sides of the last congruence:

$$\sum_{n=0}^{\infty} p_{\{3,3\}}(6n+4)q^{2n+1} \equiv 6q\frac{f_4^2 f_6^3 f_{16}^2}{f_2^8 f_8} \pmod{4}.$$

Dividing by q and replacing q^2 by q yields

$$\sum_{n=0}^{\infty} p_{\{3,3\}}(6n+4)q^n \equiv 2\frac{f_2^2 f_3^3 f_8^2}{f_1^8 f_4} \pmod{4}.$$

Thus, after some simplification, we obtain

$$\sum_{n=0}^{\infty} p_{\{3,3\}}(6n+4)q^n \equiv 2f_3^3 f_8 \pmod{4}.$$

Thanks to (2.4) and (3.3), we finally obtain

$$\sum_{n=0}^{\infty} p_{\{3,3\}}(6n+4)q^n \equiv 2\sum_{k=0}^{\infty}\sum_{l=-\infty}^{\infty}(-1)^{k+l}(2k+1)q^{3k(k+1)/2+4l(3l-1)}$$
$$\equiv 2\sum_{k=0}^{\infty}\sum_{l=-\infty}^{\infty} q^{3k(k+1)/2+4l(3l-1)} \pmod 4.$$

Now we note that the possible residues of $3k(k+1)/2$ modulo 25 are 0, 3, 5, 8, 9, 10, 13, 15, 18, 20, and 23, whereas the possible residues of $4l(3l-1)$ modulo 25 are 0, 1, 5, 6, 8, 10, 11, 15, 16, 20, and 21. Thus, a number of the form $3k(k+1)/2+4l(3l-1)$ is not congruent to $2, 7, 12$ or $22 \pmod{25}$. Therefore, the coefficients of the terms q^{25n+t}, where $t \in \{2, 7, 12, 22\}$, are congruent to 0 modulo 4, which completes the proof. □

5. Concluding remarks

As noted in Section 3, the following two congruences are direct consequences of Theorem 3.3:

$$p_{\{3,3\}}(9n+3r+2) \equiv 0 \pmod 9, \text{ for } r \in \{1,2\}.$$

In light of (3.6), a more general congruence holds, namely $p_{\{3,3\}}(3n+2) \equiv 0 \pmod 9$. This congruence can be directly derived from (1.12) in [8]. Nevertheless we note that thanks to (3.6) we can rewrite (2.1) as

$$\sum_{n=0}^{\infty} p_{\{3,3\}}(n)q^n = \frac{f_9^3}{f_3^9}\left(P(q^3)^2 + 3qf_9^3P(q^3) + 9q^2f_9^6\right).$$

Extracting the terms involving q^{3n+2}, dividing the resulting identity by q^2 and replacing q^3 by q, we are left with

$$\sum_{n=0}^{\infty} p_{\{3,3\}}(3n+2)q^n = 9\frac{f_3^9}{f_1^9},$$

which yields $p_{\{3,3\}}(3n+2) \equiv 0 \pmod 9$.

We close this work by noting that $p_{\{3,3\}}(n)$ appears to satisfy a number of Ramanujan–like congruences modulo 5. In particular, we note the following:

Conjecture 5.1. *For all $n \geq 0$,*

$$p_{\{3,3\}}(15n+6) \equiv 0 \pmod 5,$$
$$p_{\{3,3\}}(25n+6) \equiv 0 \pmod 5,$$
$$p_{\{3,3\}}(25n+16) \equiv 0 \pmod 5,$$
$$p_{\{3,3\}}(25n+21) \equiv 0 \pmod 5.$$

References

[1] B.C. Berndt, Ramanujan's Notebooks, Part III, Springer, New York, 1991.

[2] B.C. Berndt, Number Theory in the Spirit of Ramanujan. American Mathematical Society, Providence, Rhode Island (2006).

[3] E. Boll and D. Penniston, 'The 7-regular and 13-regular partition functions modulo 3', *Bull. Aust. Math. Soc.* **93** (2016), 410–419.

[4] S. Chern and L.-J. Hao, 'Congruences for partition functions related to mock theta functions', *Ramanujan J*. **48** (2019), 369–384.

[5] R. da Silva and J.A. Sellers, 'New congruences for 3-regular partitions with designated summands', *Integers* **20A** (2020), #A6

[6] R. da Silva and J.A. Sellers, 'Infinitely Many Congruences for k-Regular Partitions with Designated Summands', *Bull. Braz. Math. Soc.* **51** (2020), 357–370.

[7] D.Q. Dou, L.Guo, and B.L.S. Lin, 'Parity results for 13-regular partitions and broken 6-diamond partitions', *Ramanujan J.* **44** (2017), 521–530.

[8] D.S. Gireesh and M.S. Mahadeva Naika, 'On 3-regular partitions in 3-colors', *Indian J. Pure Appl. Math.* **50** (2019), 137–148.

[9] M.D. Hirschhorn, The power of q, a personal journey, Developments in Mathematics, vol. 49, Springer, New York (2017).

[10] M.D. Hirschhorn, 'Partitions in 3 colours', *Ramanujan J.* **45** (2018), 399–411.

[11] M. D. Hirschhorn and J. A. Sellers, 'Elementary proofs of parity results for 5-regular partitions', *Bull. Aust. Math. Soc.* **81** (2010), no. 1, 58–63.

[12] W. J. Keith, 'Congruences for m-regular partitions modulo 4', *Integers* **15A** (2015), #A11.

[13] L. Wang, 'Arithmetic properties of (k, ℓ)-regular bipartitions', *Bull. Aust. Math. Soc.* **95** (2017), no. 3, 353–364.

[14] L. Wang, 'Congruences for 5-regular partitions modulo powers of 5', *Ramanujan J.* **44** (2017), no. 2, 343–358.

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