In these notes, I will present everything we know so far about linear transformations. This material comes from sections 1.7, 1.8, 4.2, 4.5 in the book, and supplemental stuff that I talk about in class. The order of this material is slightly different from the order I used in class. We will start with the initial definition.

Definition 1 *A function $T : V \to W$ from a vector space V to a vector space W is called a linear transformation if*

$$T(u+v) = T(u) + T(v),$$
$$T(cu) = cT(u),$$

for all $u, v \in V$ and all real c.

The following facts will be used so frequently, they should be memorized as if they too were part of the definition.

Lemma 1 *If $T : V \to W$ is a linear transformation, then*

$$T(0) = 0$$

and

$$T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = c_1T(v_1) + \cdots + c_nT(v_n).$$

Associated with each linear transformation are two vector spaces:

$$\mathrm{Ker}(T) = \{v \in V \,|\, T(v) = 0\}$$

and

$$\mathrm{Range}(T) = \{T(v) \,|\, v \in V\} = \{w \in W \,|\, w = T(v) \text{ for some } v\}.$$

You should be able to verify that these two sets are, indeed, subspaces of the appropriate spaces. Make sure you know where each of these subspaces "lives." Ker(T) is a subspace of V and Range(T) is a subspace of W. If I asked you for a basis for the Range of T and you listed vectors from V instead of W, there would be a considerable penalty on an exam. As an extended example, let $T : P_3 \to M_{2\times 2}$ be defined by $T(p(t)) = p\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. What this means is that if $p(t) = at^3 + bt^2 + ct + d$ and $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, then

$$\begin{aligned} T(p(t)) &= aA^3 + bA^2 + cA + dI \\ &= a\begin{pmatrix} -2 & 2 \\ -2 & -2 \end{pmatrix} + b\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} + c\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} + d\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{1}$$

Here, I have done some computing: $A^2 = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$ and $A^3 = \begin{pmatrix} -2 & 2 \\ -2 & -2 \end{pmatrix}$. For example,

$$T(2t^3 - 3t + 1) = 2\begin{pmatrix} -2 & 2 \\ -2 & -2 \end{pmatrix} - 3\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -6 & 1 \\ -1 & -6 \end{pmatrix}.$$

We could use (1) to rewrite the definition of T:

$$T(at^3 + bt^2 + ct + d) = \begin{pmatrix} -2a + c + d & 2a + 2b + c \\ -2a - 2b - c & -2a + c + d \end{pmatrix}. \tag{2}$$

For a quick check that T is linear, it is easier to use the original definition of T:

$$\begin{aligned} T(p(t) + q(t)) &= (p + q)(A) = p(A) + q(A) = T(p(t)) + T(q(t)), \\ T(cp(t)) &= cp(A) = cT(p(t)). \end{aligned}$$

Next, let's calculate $\ker(T)$, Range(T), and bases for each. For this, it is easier to use (2) than the initial definition. For the kernel, we want polynomials $p(t)$ for which $T(p(t)) = 0$. That is, we want

$$T(at^3 + bt^2 + ct + d) = \begin{pmatrix} -2a + c + d & 2a + 2b + c \\ -2a - 2b - c & -2a + c + d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

To get the zero matrix we need $-2a + c + d = 0, 2a + 2b + c = 0$. We could put these into a matrix and row reduce, but it is easier to just let a and b be free. Subtracting the second equation from the first, $-4a - 2b + d = 0$ so $c = -2a - 2b$, $d = 4a + 2b$. The kernel of T is the set of all polynomials of the form

$$at^3 + bt^2 - (2a + 2b)t + 4a + 2b = a(t^3 - 2t + 4) + b(t^2 - 2t + 2).$$

From this, we get a basis for the kernel, $\{t^3 - 2t + 4, t^2 - 2t + 2\}$. You should check that each of these polynomials actually is in the kernel.

For the range of T, we have to answer the question "What matrices can have the form $T(p(t))$ for some polynomial $p(t)$?" Introducing some free variables, we want to know what $\begin{pmatrix} w & x \\ y & z \end{pmatrix}$ can have the form $\begin{pmatrix} -2a + c + d & 2a + 2b + c \\ -2a - 2b - c & -2a + c + d \end{pmatrix}$. We need to know what has to be true about w, x, y, z in order for a, b, c, d to exist. That is, we need to know when the system of equations $\begin{array}{ll} -2a + c + d & = w \\ 2a + 2b + c & = x \\ -2a - 2b - c & = y \\ -2a + c + d & = z \end{array}$ is consistent. We find the augmented matrix and row reduce:

$$\left(\begin{array}{cccc|c} -2 & 0 & 1 & 1 & w \\ 2 & 2 & 1 & 0 & x \\ -2 & -2 & -1 & 0 & y \\ -2 & 0 & 1 & 1 & z \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} -2 & 0 & 1 & 1 & w \\ 2 & 2 & 1 & 0 & x \\ 0 & 0 & 0 & 0 & x + y \\ 0 & 0 & 0 & 0 & -w + z \end{array}\right).$$

To have a solution, we need $x + y = 0, -w + z = 0$. That is, we have to solve yet another system of equations. In this case, we can do it by inspection: Let y and z be free, giving $x = -y, w = z$. Thus, the range consists of all matrices of the form $\begin{pmatrix} z & -y \\ y & z \end{pmatrix}$, which we can write $z\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + y\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, so a basis for the range is $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}$.

Here is a second way to calculate the range. It is based on the following theorem.

Theorem 1 *Given a linear transformation $T : V \to W$, and a basis $\{v_1, v_2, \ldots, v_n\}$ for V, then Range(T) = Span$\{T(v_1), T(v_2), \ldots, T(v_n)\}$.*

This theorem does NOT say Span$\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis, because the set could be linearly dependent. However, it does give a way to find a basis for the range: remove dependent vectors form Span$\{T(v_1), T(v_2), \ldots, T(v_n)\}$ until the set becomes independent. Once you see the proof of the Rank-Nullity theorem later in this set of notes, you should be able to prove this.

Back to our example, we first need a basis for P_3, the domain space. We might as well use the standard basis, $\{1, t, t^2, t^3\}$. Applying T to each basis vector, $\{T(1), T(t), T(t^2), T(t^3)\}$, or $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 2 \\ -2 & -2 \end{pmatrix}\right\}$ will be a spanning set for the range. The first two vectors in this set are linearly independent. However, $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = -2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and $\begin{pmatrix} -2 & 2 \\ -2 & -2 \end{pmatrix} = -4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. Since the third and fourth vectors depend on the first two, a basis is $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\right\}$. In general, this approach will produce a different basis than the first method. Also, this basis, though easier to find, is usually not as nice to work with as the one done by the first approach.

The next result is good for teachers–it gives us a way to design transformations with various properties.

Theorem 2 *Let V and W be vector spaces, and let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V. Given ANY n vectors $w_1, w_2, \ldots, w_n$, in W, there is a unique linear transformation $T : V \to W$ for which*

$$T(v_1) = w_1, \quad T(v_2) = w_2, \quad \ldots, \quad T(v_n) = w_n.$$

This theorem is sometimes expressed this way: A linear transformation is uniquely determined by its action on a basis. That is, if you know what T does to each of the vectors in some basis, then in principle, you know T.

The proof has two parts: existence, and uniqueness. That is, we must show that such a transformation exists, and then show there is at most one such transformation. For the existence, we make use of the spanning property of a basis: given any vector u in V, we can write $u = c_1v_1 + c_2v_2 + \cdots + c_nv_n$. We now define T this way: $T(u) = c_1w_1 + c_2w_2 + \cdots + c_nw_n$. That is, what ever linear combination is needed to write u as a copy of v's, use that same combination on the w's to get $T(u)$. This defines a function from V to W but it does not tell us that the function is linear. We must check this. That is, we have to check that $T(u + v) = T(u) + T(v)$ and $T(ku) = kT(u)$. So let $u = c_1v_1 + c_2v_2 + \cdots + c_nv_n$ and $v = d_1v_1 + d_2v_2 + \cdots + d_nv_n$. Then $u + v = (c_1 + d_1)v_1 + (c_2 + d_2)v_2 + \cdots + (c_n + d_n)v_n$, so

$$\begin{aligned} T(u+v) &= (c_1 + d_1)w_1 + (c_2 + d_2)w_2 + \cdots + (c_n + d_n)w_n \\ &= c_1w_1 + c_2w_2 + \cdots + c_nw_n + d_1w_1 + d_2w_2 + \cdots + d_nw_n \\ &= T(u) + T(v). \end{aligned}$$

Since $ku = kc_1v_1 + kc_2v_2 + \cdots + kc_nv_n$.

$$\begin{aligned} T(ku) &= kc_1w_1 + kc_2w_2 + \cdots + kc_nw_n \\ &= k(c_1w_1 + c_2w_2 + \cdots + c_nw_n) \\ &= kT(u), \end{aligned}$$

as desired.

Having established that there IS a linear transformation with the right property, we show there is only one. To that end, suppose $S : V \to W$ is linear and $S(v_1) = w_1, S(v_2) = w_2, \ldots, S(v_n) = w_n$. We show that S and T are the same function. First, a definition: Two functions $f(x)$ and $g(x)$ are the SAME if for all $x, f(x) = g(x)$. That is, $\sin 2x$ and $2 \sin x \cos x$ are the same function, even though they look different. So to show S and T are the same, we must show that $S(v) = T(v)$ for all vectors v in V. Given v in V, we know there are scalars $c_1, c_2, \ldots, c_n$ for which $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$. We also know that from the definition of T, $T(v) = c_1w_1 + c_2w_2 + \cdots + c_nw_n$. On the other hand,

$$\begin{aligned} S(v) &= S(c_1v_1 + c_2v_2 + \cdots + c_nv_n) \\ &= c_1S(v_1) + c_2S(v_2) + \cdots + c_nS(v_n) && \text{(by linearity)} \\ &= c_1w_1 + c_2w_2 + \cdots + c_nw_n && \text{(definition of } S\text{)} \\ &= T(v), \end{aligned}$$

as desired. Here is a proof of Theorem 10 in Chapter 1 of our book (page 72).

Theorem 3 *If $T : R^n \to R^m$ is a linear transformation, then there is a unique $m \times n$ matrix A for which $T(v) = Av$ for all v in R^n.*

This theorem says that the only linear transformations from R^n to R^m are matrix transformations. A transformation may be defined differently, but in the end, we could find an A to describe it.

Proof: We will use the previous theorem, so first we need a basis for R^n, and we may as well use the standard basis, $\{e_1, e_2, \ldots, e_n\}$. We apply the transformation, T, to each of these standard basis vectors. Suppose $T(e_1) = w_1, T(e_2) = w_2, \ldots, T(e_n) = w_n$. By the previous theorem, once we know this information, we essentially know T. Writing $v = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, since

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n,$$

we have

$$\begin{aligned} T(v) &= T(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n) \\ &= x_1 T(e_1) + x_2 T(e_2) + \cdots + x_n T(e_n) \\ &= x_1 w_1 + x_2 w_2 + \cdots + x_n w_n. \end{aligned}$$

Now let A be the matrix that has the w's as its columns: $A = (w_1 | w_2 | \cdots | w_n)$. By the way matrix multiplication works, we have

$$Av = (w_1 | w_2 | \cdots | w_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 w_1 + x_2 w_2 + \cdots + x_n w_n = T(v).$$

This completes the proof.

For an example, suppose that $T(x, y, z) = (y - z, 2x + 3y + 4z)$. Then $T(e_1) = (0, 2), T(e_2) = (1, 3), T(e_3) = (-1, 4)$. we convert these vectors to columns (we HAVE to use columns when putting things in matrix format), the matrix of the transformation is

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 3 & 4 \end{pmatrix}.$$

As a check, $\begin{pmatrix} 0 & 1 & -1 \\ 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y - z \\ 2x + 3y + 4z \end{pmatrix}$, which is $T(x, y, z)$, when put into column format instead of row format.

It is important to remember that this theorem ONLY applies to transformations from R^n to R^m. If a polynomial space, a matrix space, or even some subspace of R^n is involved, this theorem does not apply.

Here are two more examples of Theorem 2. Suppose we wish for a linear transformation from $M_{2\times 2}$ to P_2 that maps the basis vectors $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ to polynomials t^2+t+1, $t+1$, t^2+t, and t, respectively. Theorem 2 says there is a unique linear transformation that does this. Can we find a formula for it? Yes: Write the standard matrix as a linear combination of basis vectors. I will let you check that

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = (a-b)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + (b-c)\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}, +(c-d)\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} + d\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}.$$

Given this, by linearity,

$$\begin{aligned} T\begin{pmatrix}a & b\\ c & d\end{pmatrix} &= (a-b)T\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + (b-c)T\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} + (c-d)T\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} + dT\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\\ &= (a-b)(t^2+t+1) + (b-c)(t+1) + (c-d)(t^2+t) + dt\\ &= (a-b+c-d)t^2 + at + (a-c). \end{aligned}$$

Suppose, instead, we want a linear transformation from $M_{2\times 2}$ to P_2 with kernel equal to the span of $\left\{\begin{pmatrix}1 & 2\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 2 & 1\end{pmatrix}\right\}$ and range the span of $\{t+1, t^2+t\}$. One has to be careful with these problems, as some combinations are not possible. In this case, such a transformation exists, and their are infinitely many such transformations. How do we find such a transformation? First, we extend $\left\{\begin{pmatrix}1 & 2\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 2 & 1\end{pmatrix}\right\}$ to a basis for all of $M_{2\times 2}$. Here is one such basis: $\left\{\begin{pmatrix}1 & 2\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 2 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}\right\}$. Theorem 2 says there is a (unique) transformation that maps a basis onto any given set of vectors of the same size. We want the first two vectors to map to 0, since they must be in the kernel. We map the other two vectors to the two basis vectors for the range. That is, we look for a transformation that does this:

$$T\begin{pmatrix}1 & 2\\ 0 & 0\end{pmatrix} = 0, \quad T\begin{pmatrix}0 & 0\\ 2 & 1\end{pmatrix} = 0, \quad T\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} = t+1, \quad T\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix} = t^2+t.$$

We now proceed as in the previous example. We write a general matrix as a combination of basis vectors, and then apply T to get a formula. We have

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = (b-a)\begin{pmatrix}1 & 2\\ 0 & 0\end{pmatrix} + (c-d)\begin{pmatrix}0 & 0\\ 2 & 1\end{pmatrix} + (2a-b)\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} + (2d-c)\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix},$$

so

$$\begin{aligned} T\begin{pmatrix}a & b\\ c & d\end{pmatrix} &= (b-a)T\begin{pmatrix}1 & 2\\ 0 & 0\end{pmatrix} + (c-d)T\begin{pmatrix}0 & 0\\ 2 & 1\end{pmatrix} + (2a-b)T\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} + (2d-c)T\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}\\ &= (b-c)(0) + (c-d)(0) + (2a-b)(t+1) + (2d-c)(t^2+t)\\ &= (2d-c)t^2 + (2a-b-c+2d)t + (2a-b). \end{aligned}$$

Answers like this can be checked. If we wanted the kernel, we would look for matrices that get sent to 0 so we need $2a - b = 0,\ 2d - c = 0,\ 2a - b - c + 2d = 0$. The last equation is just the sum of the first two, so we need $b = 2a$ and $c = 2d$, which quickly tells us we have the right kernel. I will let you check that the range also works.

The Rank-Nullity Theorem

We had a theorem for matrices called the Rank-Nullity theorem. It stated that the dimension of the column space + the dimension of the null space of a matrix is n, the number of columns in the matrix. We now extend this result to linear transformations.

Theorem 4 *(The Rank-Nullity Theorem) Let $T : V \to W$ be a linear transformation from a finite dimensional vectors space V to a vector space W. Then*

$$dim(Ker(T)) + dim(Range(T)) = dim\ V.$$

Proof: As with almost every proof that involves dimensions, we make use of bases for various vector spaces involved. What we must do is relate the sizes of bases for Ker(T), Range(T), and V. Since Ker(T) is a subspace of V, it makes sense to start with these two vector spaces (Ker(T), and V). Let $\{u_1, u_2, \ldots, u_k\}$ be a basis for Ker(T) (the smaller of the two spaces). This is an independent set of vectors in V, so we can extend it to a basis for all of V. Let this extended basis be $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_m\}$. In forming these two bases, we have labeled the dimensions of Ker(T) and V. That is, we have said that $\dim(\text{Ker}(T)) = k$, and $\dim(V) = k + m$. This means that we must show that the dimension of the range of T is m.

Claim A basis for the range of T is $\{T(v_1), T(v_2), \ldots, T(v_m)\}$. If we can verify this claim, we will have finished the proof. To show this set is a basis, we must establish both the spanning property and the independence property. We tackle these properties in the order listed.

Spanning: Let w be in the range of T. This means that $w = T(v)$ for some v in V. This v can be written as a combination of basis vectors so

$$v = c_1u_1 + \cdots + c_ku_k + d_1v_1 + \cdots + d_mv_m.$$

Applying the transformation and making use of linearity,

$$\begin{aligned} w = T(v) &= T(c_1u_1 + \cdots + c_ku_k + d_1v_1 + \cdots + d_mv_m \\ &= c_1T(u_1) + \cdots + c_kT(u_k) + d_1T(v_1) + \cdots + d_mT(v_m). \end{aligned}$$

Since the u's are all in the kernel of T, we have $T(u_j) = 0$ for each j. Consequently,

$$w = d_1T(v_1) + \cdots + d_mT(v_m),$$

which shows that w is in the span of $\{T(v_1), T(v_2), \ldots, T(v_m)\}$.

Independence: Suppose that $d_1T(v_1) + \cdots + d_mT(v_m) = 0$ for some scalars $d_1, \ldots, d_m$. We must show that all the d's are forced to be 0. By linearity (in the opposite direction we usually use it), $d_1T(v_1)+\cdots+d_mT(v_m) = 0 \to T(d_1v_1+\cdots+d_mv_m) = 0$, so $d_1v_1+\cdots+d_mv_m$ is in the kernel of T. Since we have a basis for $\mathrm{Ker}(T)$, we have

$$d_1v_1 + \cdots + d_mv_m = c_1u_1 + \cdots + c_ku_k.$$

This looks like a dependence relation among the u's and v's, but the u's and v's are independent. The only possibility, then, is that all the coefficients, all the c's and all the d's are zero. In particular, all the d's must be zero. This shows that $\{T(v_1), T(v_2), \ldots, T(v_m)\}$ is a linearly independent set, completing the proof that it is a basis.

We now have a new way to find a basis for the range of T. For example, going back to our first transformation, $T(at^3 + bt^2 + ct + d) = \begin{pmatrix} -2a+c+d & 2a+2b+c \\ -2a-2b-c & -2a+c+d \end{pmatrix}$, recall that $\mathrm{Ker}(T)$ has a basis of $\{t^2-2t+2, t^3-2t+4\}$. We extend this to a basis for P_3. In fact, such a basis is $\{t^2-2t+2, t^3-2t+4, 1, t\}$. One way to see this is a basis is to show that each of the standard basis vectors, $1, t, t^2, t^3$ are in the span of the set. You should check this. By the proof of the Rank-Nullity Theorem, if we delete the vectors in the kernel of T, leaving us with $\{1, t\}$, and then apply T to each of these vectors, we should get a basis for the range. That is, a basis for the range of T is

$$\{T(1), T(t)\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \right\}.$$

Here is another example. Let $T : P_3 \to P_3$ be defined by

$$T(p(t)) = p(t+2) - (t+1)p'(t).$$

For example,

$$\begin{aligned} T(t^3-2t+3) &= (t+2)^3 - 2(t+2) + 3 - (t+1)(3t^2-2) \\ &= t^3 + 6t^2 + 12t + 8 - 2t - 4 + 3 - 3t^3 - 3t^2 + 2t + 2 \\ &= -2t^3 + 3t^2 - 14t + 9. \end{aligned}$$

You should verify that T is a linear transformation. We will find bases for the kernel and range of T. For the kernel, we want those polynomials, $p(t)$ with $T(p(t)) = 0$. Letting $p(t) = at^3 + bt^2 + ct + d$, we have

$$\begin{aligned} T(p(t)) &= a(t+2)^3 + b(t+2)^2 + c(t+2) + d - (t+1)(3at^2 + 2bt + c) \\ &= -2at^3 + (3a-b)t^2 + (12a+2b)t + (8a+4b+c+d). \end{aligned}$$

For this to be 0, we need $-2a = 0, 3a-b = 0, 12a+2b = 0, 8a+4b+c+d = 0$, which quickly reduces to $a = 0, b = 0, c+d = 0$. Writing $d = -c$, the kernel consists of all polynomials $p(t) = ct - c = c(t-1)$, so the kernel is one-dimensional and a basis is $\{t-1\}$.

For the range, we extend $\{t-1\}$ to a basis for P_3. As usual, we append the standard basis vectors to the set: $\{t-1, 1, t, t^2, t^3\}$ is a spanning set for P_3. Next, we remove dependent vectors from among $1, t, t^2, t^3$. In fact, we need only remove t to get $\{1-t, 1, t^2, t^3\}$. (We know we need exactly four vectors, so there could only be one dependence.) Next, by the proof of the Rank-Nullity Theorem, a basis for the range is $\{T(1), T(t^2), T(t^3)\}$. Applying T, we get our basis

$$\{1, -t^2+2t+4, -2t^3+3t^2+12t+8\}.$$

We could have found a basis for the range without using the Rank-Nullity Theorem. We would look for polynomials et^3+ft^2+gt+h that could equal $T(p(t))$ for some polynomial $p(t)$. We would get the system

$$\begin{aligned} -2a &= e \\ 3a-b &= f \\ 12a+2b &= g \\ 8a+4b+c+d &= f. \end{aligned}$$

We need this system to be consistent. This leads to a row reduction:

$$\left(\begin{array}{cccc|c} -2 & 0 & 0 & 0 & \text{e} \\ 3 & -1 & 0 & 0 & \text{f} \\ 12 & 2 & 0 & 0 & \text{g} \\ 8 & 4 & 1 & 1 & \text{h} \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} -2 & 0 & 0 & 0 & \text{e} \\ 3 & -1 & 0 & 0 & \text{f} \\ 18 & 0 & 0 & 0 & \text{2f + g} \\ 8 & 4 & 1 & 1 & \text{h} \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} -2 & 0 & 0 & 0 & \text{e} \\ 3 & -1 & 0 & 0 & \text{f} \\ 0 & 0 & 0 & 0 & \text{9e + 2f + g} \\ 8 & 4 & 1 & 1 & \text{h} \end{array}\right).$$

I will let you check that the only relation we need for consistency is $9e+2f+g=0$. Writing this $g=-9e-2f$, we have range polynomials of the form $et^3+ft^2-(9e+2f)t+h = e(t^3-9t)+f(t^2-2t)+h\cdot 1$, leading to the basis

$$\{t^3-9t, t^2-2t, 1\}.$$

This is different from the previous basis, and this is usually the case. The basis produced by the Rank-Nullity Theorem is usually easier to get, but the direct approach, though more work intensive, usually produces a nicer basis. For example, suppose we wanted to know if t^3-2t^2-5t-1 is in the range of T. It is easier to try to write this polynomial as a combination of $t^3-9t, t^2-2t, 1$ than to use the other basis. We would write

$$t^3-2t^2-5t-1 = a(t^3-9t)+b(t^2-2t)+c\cdot 1,$$

which quickly gives $a=1, b=-2, c=1$. We need to check that this works (which involves checking that the coefficient of t is correct), but this also is easy. Thus, this polynomial IS in the range of T. To use the other basis, we would need

$$t^3-2t^2-5t-1 = a\cdot 1+b(-t^2+2t+4)+c(-2t^3+3t^2+12t+8).$$

To find a, b, c here requires solving (an easy) system, and checking that it is consistent takes a little more work as well.