An introduction to the normalized Laplacian

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$$A = \{2, -1, -1\}$$

$+$	2	-1	-1
2	4	1	1
-1	1	-2	-2
-1	1	-2	-2

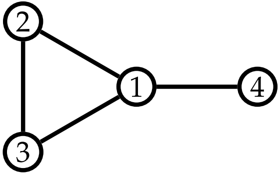
$*$	2	-1	-1
2	4	-2	-2
-1	-2	1	1
-1	-2	1	1

Are there any other examples where $A + A = A * A$ (as a multi-set)?
Yes! $A = \{0, 0, \ldots, 0\}$ or $A = \{2, 2, \ldots, 2\}$

Are there any other nontrivial examples?

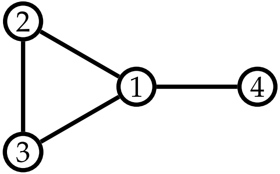
What does this have to do with this talk?

Graphs are collections of objects (vertices) and relations between them (edges).



Graphs are very universal and can model just about everything.

Graphs are collections of objects (vertices) and relations between them (edges).

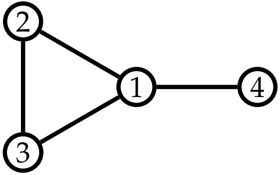


Graphs are very universal and can model just about everything.

Matrices are arrays of numbers

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Graphs are collections of objects (vertices) and relations between them (edges).



Graphs are very universal and can model just about everything.

Matrices are arrays of numbers with benefits.

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Example: eigenvalues are λ where for some $\mathbf{x} \neq 0$ we have $A\mathbf{x} = \lambda\mathbf{x}$.

$\{2.17..., 0.31..., -1, -1.48...\}$

GRAPH $\longleftrightarrow$ MATRIX $\longrightarrow$ EIGENVALUES

$\longleftarrow$

Spectral Graph Theory

What are relationships between the structure of a given graph and the eigenvalues of a matrix associated with the graph.

Common matrices

- Adjacency, A: Matrix indicates which vertices are adjacent. Eigenvalues can count closed walks, so can count edges and test bipartite-ness.
- Laplacian, $L = D - A$: Derived from signed incidence matrix, is positive semi-definite. Eigenvalues can count edges and number of components.
- Signless Laplacian, $Q = D + A$: Derived from unsigned incidence matrix, is positive semi-definite. Eigenvalues can count edges and number of *bipartite* components.

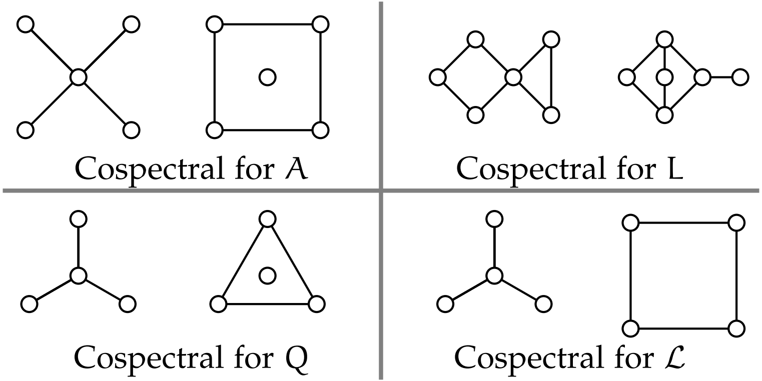
Little less common matrix

- Normalized Laplacian, $\mathcal{L} \text{``}=\text{''} D^{-1/2}(D - A)D^{-1/2}$: Normalizes the Laplacian matrix, and is tied to the probability transition matrix.
  - Eigenvalues lie in the interval $[0, 2]$.
  - Multiplicity of 0 is number of components.
  - Multiplicity of 2 is number of bipartite components.
  - Tests for bipartite-ness.
  - *Cannot* always detect number of edges.

Summarizing

	bip.	# comp.	# bip. comp.	# edges
A	YES	no	no	YES
L	no	YES	no	YES
Q	no	no	YES	YES
$\mathcal{L}$	YES	YES	YES	no

Cospectral pairs



Normalizing is good!

Theorem $|\mathbf{x}^*B\mathbf{y}| \leq \sigma(B)\|\mathbf{x}\|\|\mathbf{y}\|$.

Discrepancy for a d-regular graph is the minimum α so

$$\left| e(X, Y) - \frac{d}{n}|X||Y| \right| \leq \alpha\sqrt{|X||Y|}.$$

(Discrepancy measures randomness of edge placement.)

Theorem (Alon-Chung)

Discrepancy is bounded above by $\sigma_2(A)$.

Proof. Set $\mathbf{x} = 1_X$, $\mathbf{y} = 1_Y$, and $B = A - \frac{d}{n}J$ (i.e., subtract out largest eigenvalue) and use preceding result.

Normalizing is good!

What about non-regular graphs?

`funnycatpix.com` `google.com`

A has a vertex-centric measure of size. $\mathcal{L}$ has an edge-centric measure of size ($\mathrm{vol}(X) = \sum_{v \in X} d_v$).

Theorem (Chung) For G a graph and $X, Y \subseteq V$:

$$\left| e(X, Y) - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)} \right| \leq \sigma_2(D^{-1/2} A D^{-1/2}) \sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)}.$$

Proof. Set $\mathbf{x} = D^{1/2} 1_X$, $\mathbf{y} = D^{1/2} 1_Y$, and $B = D^{-1/2} A D^{-1/2} - \frac{1}{\mathrm{vol}(G)} D^{1/2} J D^{1/2}$ (i.e., subtract out largest eigenvalue) and use preceding result.

Using weighted graphs

Let $w(u, v)$ be the weight of the edge between u and v and $d(u) = \sum_v w(u, v)$. If we let $A_{u,v} = w(u, v)$ and $D(u, u) = d(u)$, then $\mathcal{L} = D^{-1/2}(D - A)D^{-1/2}$.

Proposition

Let G be a weighted graph and αG the weighted graph resulting from scaling the weight of each edge by α. Then $\mathcal{L}_G = \mathcal{L}_{\alpha G}$.

Proof: Scaling terms cancel out by the normalization.

Harmonic eigenvectors

If $A\mathbf{y} = \mu\mathbf{y}$, then at each vertex t we have

$$\sum_{v} w(t, v)\mathbf{y}(v) = \mu\mathbf{y}(t).$$

Let $\mathcal{L}\mathbf{x} = \lambda\mathbf{x}$ and let $\mathbf{x} = D^{1/2}\mathbf{y}$ ($\mathbf{y}$ is known as the harmonic eigenvector). Then $(D - A)\mathbf{y} = \lambda\mathbf{y}$, or $A\mathbf{y} = (1 - \lambda)D\mathbf{y}$, or at each vertex t we have

$$\sum_{v} w(t, v)\mathbf{y}(v) = (1 - \lambda)d(t)\mathbf{y}(t).$$

$\mathbf{1}$ is harmonic eigenvector for $\lambda = 0$. All other harmonic eigenvectors $\mathbf{y}$ satisfy $\sum d(u)\mathbf{y}(u) = 0$.

Graph operations

The *Cartesian product* of G and H, denoted $G \square H$, the *tensor product* of G and H, denoted $G \times H$, and the *strong product* of G and H, denoted $G \boxtimes H$, all have vertex set $\{(a, b) : a \in V(G), b \in V(H)\}$ and edge sets respectively as follows:

$$E(G \square H) = \big\{\{(a, b), (c, d)\} : a{=}c \text{ and } b{\sim}d; \text{ or } a{\sim}c \text{ and } b{=}d\big\}$$

$$E(G \times H) = \big\{\{(a, b), (c, d)\} : a{\sim}c \text{ and } b{\sim}d\big\}$$

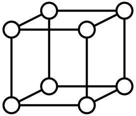
$$E(G \boxtimes H) = E(G \square H) \cup E(G \times H).$$

The *join* of G and H, denoted $G \vee H$, is the graph formed by taking the disjoint union of G and H and then adding an edge between every vertex in G and every vertex in H.

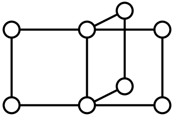
$\mathcal{L}$ and graph operations

Proposition

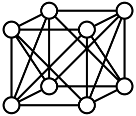
There is *no* way to determine the spectrum of $\mathcal{L}$ for the graphs $G \square H$, $G \boxtimes H$ or $G \vee H$ by only knowing the spectrum of $\mathcal{L}$ of G and H.



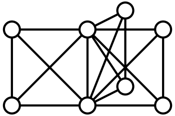
(a) $K_{2,2} \square K_2$



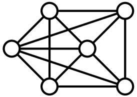
(b) $K_{1,3} \square K_2$



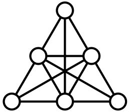
(c) $K_{2,2} \boxtimes K_2$



(d) $K_{1,3} \boxtimes K_2$



(e) $K_{2,2} \vee K_2$



(f) $K_{1,3} \vee K_2$

$\mathcal{L}$ and graph operations

If G and H are regular then we can easily compute eigenvalues of $G \,\square\, H$ and $G \boxtimes H$ using known facts about adjacency matrix. (For regular graphs we can easily go from the spectrum of one matrix to another.)

Theorem

Eigenvalues of $G \times H$ *are* $\{\lambda + \mu - \lambda\mu\}$ *where* λ *ranges over all eigenvalues of* G *and* μ *ranges over all eigenvalues of* H.

$\mathcal{L}$ and joins

Theorem

Let G *be an* r*-regular graph on* n *vertices with eigenvalues* $\{\lambda_i\}$ *and let* H *be an* s*-regular graph on* m *vertices with eigenvalues* $\{\theta_j\}$*. Then the eigenvalues of* $G \vee H$ *are*

$$\left\{0, 2 - \frac{r}{m+r} - \frac{s}{n+s}\right\} \cup \left\{\frac{m + r\lambda_i}{m+r}\right\} \cup \left\{\frac{n + s\theta_j}{n+s}\right\}.$$

General idea

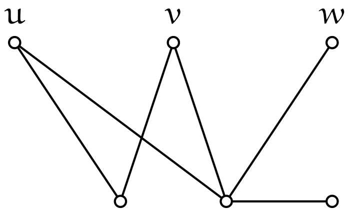
When a “regular” subgraph *cones* to the rest of the graph, we can extract eigenvalues from the subgraph and then collapse it to a vertex.

Twin vertices

In a weighted graph, two (non-adjacent) vertices u and v are twins if for every vertex x

$$w(u, x) = w(v, x).$$

(In simple graphs, this translates as having the same neighbors.)



Über vertices

In a weighted graph a pair of twins can be used to construct (harmonic) eigenvector for eigenvalue 1. Suppose that u and v are twins then

$$\mathbf{y} = \mathbf{1}_u - \mathbf{1}_v$$

satisfies at each vertex t

$$\sum_v w(t, v)\mathbf{y}(v) = 0 = (1 - 1)d(t)\mathbf{y}(t).$$

Observation

Remaining eigenvectors have same values at u and v. So treat u and v as a single über-vertex uv (edge weights summed).

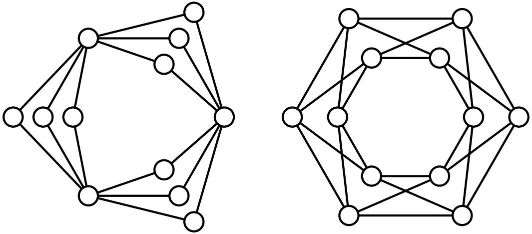
Collapsing twins

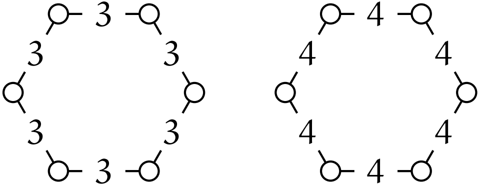
- A set of k mutual twins will contribute 1 to the spectrum $k-1$ times. We then "collapse" the set of k vertices into a single über-vertex and appropriately adjust the edge weights.
- After collapsing each set of twins we now have a weighted graph and the remaining eigenvalues are found using this smaller graph.

Theorem

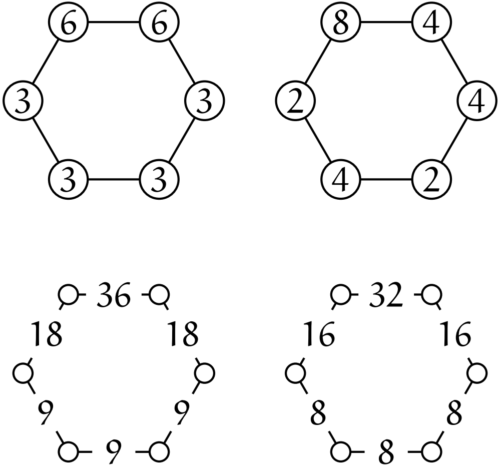
If two graphs collapse to graphs which differ by scaling and in each graph we removed the same number of "twins", then the graphs are cospectral.

Example



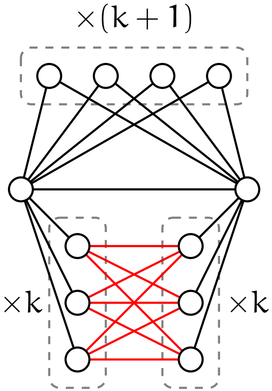


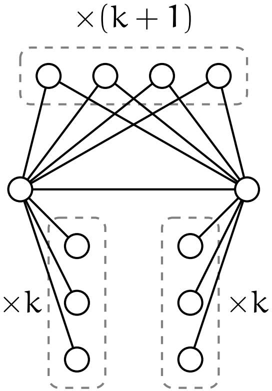
Another example



Cospectral with subgraph!

With differing number of edges it is theoretically possible to be cospectral with a subgraph.





After removing twins

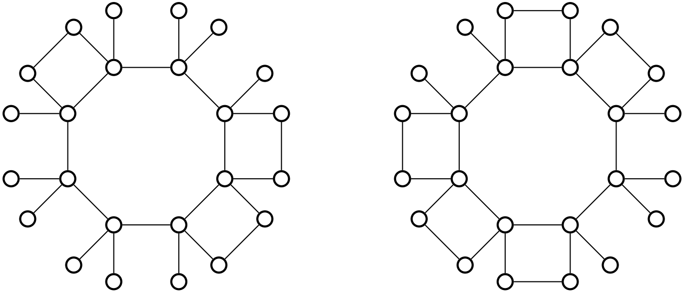
Not just simple rescaling!

Characteristic polynomial for both graphs is:

$$x^5 - \frac{6k^2 + 8k + 3}{4(k+1)^2}x^3 - \frac{1}{4(k+1)}x^2 + \frac{k(2k+1)}{4(k+1)^2}x$$

A construction

For any circular word in P and C we can form a graph by gluing in a circular chain P_4's and C_4's. For example PPCCPPPC and CCPPCCCP are:



Theorem (B-Heysse) Changing *all* P↔C does not effect the spectrum of $\mathcal{L}$

Kemeny's constant and $\mathcal{L}$

Kemeny's constant, denoted $K(G)$, is the expected first passage time from an unknown starting point to an unknown destination point.

Theorem (Levene-Loizou)

If $\rho_{n-1} \leq \cdots \leq \rho_1 < \rho_0 = 1$ *are eigenvalues of transition matrix* $D^{-1}A$ *then*

$$K(G) = \sum_{i=1}^{n-1} \frac{1}{1-\rho_i}.$$

If $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_{n-1}$ are eigenvalues of $\mathcal{L}$ then $K(G) = \sum_{i=1}^{n-1} \frac{1}{\lambda_i} = -\frac{c_2}{c_1}$

where $p_{\mathcal{L}}(t) = t^n + \cdots + c_2 t^2 + c_1 t$.

Kemeny's constant and $\mathcal{L}$

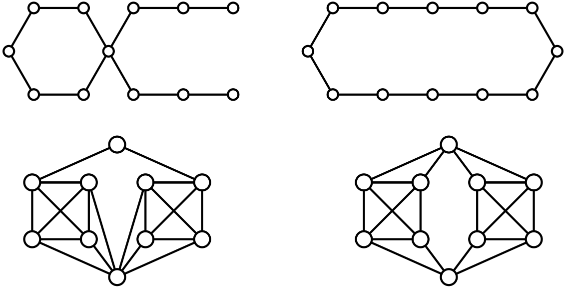
To compute Kemeny's constant it suffices to compute the last two nonzero terms of characteristic polynomial of $\mathcal{L}$. In some cases this can be done by collapsing twins and using recursion.

Let G be the graph on $n = 3\ell$ vertices consisting of two cliques on ℓ vertices joined by a path on ℓ vertices. Then

$$\begin{aligned} K(G) &= \frac{n^6 + 3n^5 + 69n^4 - 243n^3 + 837n^2 - 3159n + 5832}{54(n^3 + 9n)} \\ &= \frac{(1 + o(1))n^3}{54}. \end{aligned}$$

Still much to do!

Many things are still unknown about the normalized Laplacian. In addition to having differing number of edges, it is possible for regular graphs to be cospectral with non-regular graphs.



$A + A = A * A$

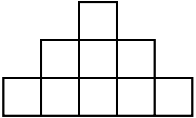
$$\left\{2, \frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}, \frac{-\sqrt{5}-1}{2}, \frac{-\sqrt{5}-1}{2}\right\}$$

- For the adjacency matrix the eigenvalues of $G \square G$ are all possible sums of eigenvalues.
- For the adjacency matrix the eigenvalues of $G \times G$ are all possible products of eigenvalues.
- $C_{2k+1} \square C_{2k+1} \cong C_{2k+1} \times C_{2k+1}$.
- Take A to be the eigenvalues of the adjacency matrix for an odd cycle:

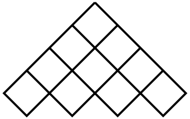
$$A = \left\{2\cos\frac{2\pi k}{2k+1} : k = 0, 1, \ldots, 2k\right\}.$$

Which is bigger?

The number of ways to build the following pyramid with any combination of $1 \times k$ or $k \times 1$ rectangles.



The number of ways to color the following pyramid white and green so that no two green squares share an edge.



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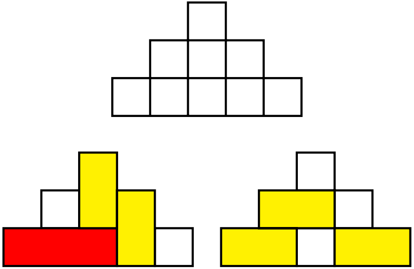

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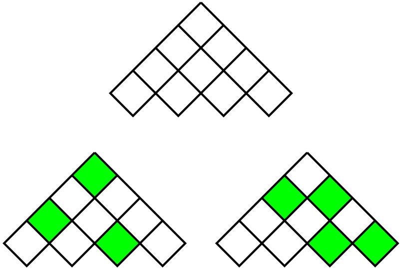
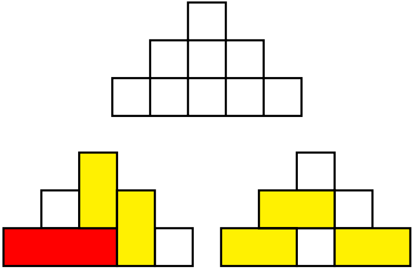


Which is bigger? They're equal!

The number of ways to build the following pyramid with any combination of $1\times k$ or $k\times 1$ rectangles.



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