Econ 3023 Microeconomic Analysis

Chapter 8: Slutsky Decomposition

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Introduction
Decomposing Effects
Giffen is Income-Inferior
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Overview

Chpt 6: Categories and definitions.
Chpt 8: Price change → demand.
Chpt 14: Price change → welfare.

Discussion 1.1 (Orange vs Tomato)

- Listen to NPR News Clip
  Orange juice is not a required staple for people to live. So they will turn to other things and that will bring down prices eventually.
- When the price of OJ goes up, do people
  1. increase consumption of tomato juice to substitute away from OJ, or
  2. can they possibly consume less tomato juice?

Exogenous Variable

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Overview

- $\Delta x_c$ denotes change in $x_c$.
- If $x_c = 5$ and then $x_c$ becomes 2 after the price change (or income change), $\Delta x_c = -3$.
- $\Delta x$ denotes change in $x_T$.
- If $x_T = 3$ and then $x_T$ becomes 8 after the price change (or income change),
  $$\Delta x = \begin{bmatrix} -3 \\ 5 \end{bmatrix}.$$

Split the price change in three progressive stages:
1. **Original**: $x^O$
2. **Transitional**: $x^T$
3. **Final**: $x^F$

Background

- Consider $(p^O_C, p_T) = (100, 1)$ and $(p^F_C, p_T) = (1, 1)$.
- $\Delta x_c$ has two components:
  1. **Income effect**, $x_\triangleleft^C$: some money left after purchasing $x^C_T$.
  2. **Substitution effect**, $x_\triangleleft^C$: Liz has to give up only one slice for a cup of tea.
x\textsuperscript{•} responds to the change in p\textsubscript{C} as well:

- **Income effect**, x\textsuperscript{•} ▲
- **Substitution effect**, x\textsuperscript{•} ▼
- In total, x\textsuperscript{•} ▲ or ▼?

---

**Definition 1.2 (Income & Substitution Effect)**

- **Income effect** is a change in demand \( \Delta x\textsuperscript{i} \) due to having more purchasing power.
- **Substitution effect** is a change in demand \( \Delta x\textsuperscript{S} \) due to change in relative price.
- **Total effect** \( \Delta x \) is the sum of the above:
  \[
  \Delta x := \Delta x\textsuperscript{i} + \Delta x\textsuperscript{S}.
  \]

---

To resolve ambiguity with tomato juice consumption in Discussion 1.1, we want to separate two effects from each other. Recall the change in parameters:

- Income change:
- Price change:
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Decomposing Effects
- Eliminating Income Effect
- Slutsky Decomposition
- Example 1: Cobb-Douglas

Giffen is Income-Inferior

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Eliminating Income Effect

- Goal: Eliminate the income effect (change in purchasing power) and isolate the substitution effect.
- Idea: You can compensate the income to suppress the change in purchasing power.
- Then let Liz solve her UMP with compensated income.
Eliminating Income Effect

\[ x_C, x_T \]

\[ BL^O, BL^T, BL^F \]
Eliminating Income Effect

- Adjust (compensate) Liz’s income so that she can still buy her original bundle $x^0$ under the new price $(p_C^F, p_T)$.
- Give her a hypothetical budget $m^T := p_C^F x_C^0 + p_T x_T^0$.
- Notice the difference:
  $$m^T = p_C^F x_C^0 + p_T x_T^0 < p_C^0 x_C^0 + p_T x_T^0 = m.$$
- $x^0$ just uses up $m$ under $p = (p_C^0, p_T)$.
- The same $x^0$ costs less ($m^T < m$) under $p = (p_C^F, p_T)$.

We'll take away $m - m^T$ dollars from $m$.
- Liz: I don't feel any change in my purchasing power. With income adjustment, I still use up my income to purchase $x^0$ regardless of the price.
- Or equivalently, change $(p_C^0, p_T, m) \rightarrow (p_C^F, p_T, m^T)$ reserves Liz’s purchasing power since she can afford $x^0$ in both environments.
- Setting income at $m^T$ suppresses the income effect and isolates the substitution effect.

We solve two problems and compare the result:

1. **UMP**:
   $$\max u(x_C, x_T) \text{ s.t. } p_C^0 x_C + p_T x_T = m$$
   with the solution
   $$x^0 = \begin{bmatrix} \varphi^C (p_C^0, p_T, m) \\ \varphi^T (p_C^0, p_T, m) \end{bmatrix}.$$

2. **Transition: UMP**
   $$\max u(x_C, x_T) \text{ s.t. } p_C^F x_C + p_T x_T = m$$
   with the solution
   $$x^F = \begin{bmatrix} \varphi^C (p_C^F, p_T, m) \\ \varphi^T (p_C^F, p_T, m) \end{bmatrix}.$$
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Eliminating Income Effect

Problem 2.1 (UMP)
- \( \text{max} (x, C, T) \) s.t. \( pO C + pT T = m \)
- \( x^D = (\phi(pO C, pT, m), \phi(pO C, pT, m)) \).
- ↓ pure substitution effect \( \Delta x^S \)

Problem 2.2 (UMP)
- \( \text{max} (x, C, T) \) s.t. \( pO C + pT T = m^T \)
- \( x^T = (\phi(pO C, pT, m), \phi(pO C, pT, m^T)) \).
- \( m^T = pO C + pT T \).
- ↓ pure income effect \( \Delta x^I \)

Problem 2.3 (UMP)
- \( \text{max} (x, C, T) \) s.t. \( pF C + pT T = m^F \)
- \( x^F = (\phi(pF C, pT, m), \phi(pF C, pT, m^F)) \).
- \( m^F = pF C + pT T \).
- \( \Delta x = \Delta x^I + \Delta x^S \).

Slutsky Decomposition

- Substitution effect \( \Delta x^S \) is given by
  \[ \Delta x^S = \phi(p^F C, pT, m^T) - \phi(pO C, pT, m) \].
- Income effect \( \Delta x^I \) is given by
  \[ \Delta x^I = \phi(p^F C, pT, m) - \phi(p^F C, pT, m^T) \].
- Total effect \( \Delta x \) is given by
  \[ \Delta x = \phi(p^F C, pT, m) - \phi(p^F C, pT, m^T) = \Delta x^I + \Delta x^S \].

Definition 2.4 (Slutsky Decomposition)

Slutsky decomposition splits the total effect into two parts:
\[ \Delta x = \Delta x^I + \Delta x^S \].
Example 2.5 (Cobb-Douglas Utility Function)

Suppose Liz consumes cheesecakes $x_C$ and tea $x_T$. Initial price of $p^0_C = 2$ was slashed in half to $p^F_C = 1$. $p_T = 1$ and $m = 16$ throughout. Her preference is represented by

$$u(x_C, x_T) = x_C x_T,$$

whose MRS at $(x_C, x_T)$ is $\frac{-x_T}{x_C}$. What are $\Delta x^S$ and $\Delta x^f$?

Steps:

1. **UMP**: Find $x^0$.
2. Find $m^T := p^0_C x^0 + p_T x_T$.
3. **UMP**: Find $x^T$.
4. **UMP**: Find $x^f$.
5. Compute $\Delta x := \Delta x^f + \Delta x^S$. 

Example 1: Cobb-Douglas

- **UMP**:
  
  $$\max u(x_C, x_T) = x_C x_T \quad \text{s.t.} \quad p^0_C x_C + x_T = m.$$  

- MRS at $(x_C, x_T)$ is $\frac{-x_T}{x_C}$.
- $x^0 = (4, 8)$. 

---
Example 1: Cobb-Douglas

Find \( m^T \).
\[
\begin{align*}
\mathbf{x}^O &= (4, 8), \\
m^T &= \mathbf{p}^T \mathbf{x}^O = 4 + 8 = 12.
\end{align*}
\]

\( \text{UMP}^T: \)

\[
\max u(x_C, x_T) = x_C x_T \text{ s.t. } \mathbf{p}^T x_C + x_T = m^T.
\]
\[
\begin{align*}
\mathbf{x}^T &= (6, 6), \\
\Delta \mathbf{x} &= \mathbf{x}^T - \mathbf{x}^O = \begin{pmatrix} 6 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}.
\end{align*}
\]
Example 1: Cobb-Douglas

- UMP:
  \[ \max_u(x_C, x_T) = x_C x_T \quad \text{s.t.} \quad p_C x_C + x_T = m. \]
  \[ x^F = (8, 8). \]
  \[ \Delta x^F = x^F - x^T = \left( \begin{array}{c} 8 \\ 8 \end{array} \right) - \left( \begin{array}{c} 6 \\ 6 \end{array} \right) = \left( \begin{array}{c} 2 \\ 2 \end{array} \right). \]

- Slutsky decomposition:
  \[ \Delta x = \Delta x^d + \Delta x^s = \left( \begin{array}{c} 2 \\ 2 \end{array} \right) + \left( \begin{array}{c} 2 \\ -2 \end{array} \right) = \left( \begin{array}{c} 4 \\ 0 \end{array} \right). \]
Introduction

Decomposing Effects

Giffen is Income-Inferior

- Fact
  - Normal Good
  - Income-Inferior Good

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Now We Know

Fact

Fact 3.1 (Sign of Substitution Effect)
Substitution effect $\Delta x_C^+$ against $p_C^-$ is positive if preferences are convex.

- If purchasing power remains the same, decrease in $p_C$ results in increase in $x_C$.\(^1\)

\(^1\)Note

- LOD says $p_C^-$ implies $x_C^+.$
- The fact above says $p_C^-$ implies $x_C^+$ if the purchasing power remains the same.

Normal Good

- For a normal good with convexity,
Suppose $p_C \downarrow$.

Slutsky decomposition:

$$\Delta x_C = \Delta x_C^S + \Delta x_C^F.$$  
+ by normality + by Fact 3.1.

Conclude: $p_C \downarrow \Rightarrow \Delta x_C ^> 0$.

Conclude: a normal good satisfies LOD.

For an income-inferior good with convexity,
Decomposing Effects

Giffen is Income-Inferior

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**Income-Inferior Good**

- Suppose \( p_C \downarrow \).
- Slutsky decomposition:
  \[
  \Delta x_C = \Delta x_C^I + \Delta x_C^S .
  \]
  - by inferiority
  - by Fact 3.1
- Can **not** conclude: \( p_C \downarrow \Rightarrow \Delta x_C \uparrow \).  
- Can **not** conclude: an income-inferior good satisfies LOD

In case of extreme income-inferiority, income effect may be **larger** in magnitude than the substitution effect.
- causing \( x_C \downarrow \) with \( p_C \downarrow \).
- Conclude: an income-inferior cheesecake may be Giffen.
- Recall the definition: Giffen good: \( p_C \downarrow \Rightarrow x_C \downarrow \).
Conversely, if a cheesecake is Giffen,
\[ \Delta x^c = \Delta x^c_C + \Delta x^C, \]
\[ \Delta x^c_C \text{ has to be } \]
Giffen cheesecake has to be 
income-inferior.

Proof.
See above.

Remark
There is no such thing as a normal Giffen cheesecake.
### Intro

### Decomposing Effects

### Giffen is Income-Inferior

#### Examples
- Example 2: Perfect Complements
- Example 3: Perfect Substitutes
- Example 4: Quasi-Linear Preferences

### Hicks

#### Now We Know

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**Example 2: Perfect Complements**

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<td>BL^F</td>
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Rotation does not change $x^*$: $\Delta x^* = x^T - x^O = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$\Delta x = \Delta x^T + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
Example 3: Perfect Substitutes

- $x^T$ is already $x^F$.
- $\Delta x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \Delta x^S$ (all the change is due to substitution effect).

Example 4: Quasi-Linear Preferences
Example 4: Quasi-Linear Preferences

- **MRS is constant at any given** $x_T$: $\text{MRS}(1, 2) = \text{MRS}(30, 2) = \text{MRS}(32170984872109874, 2)$.
- i.e., one basket is worth the same amount of tea regardless of the number of baskets.
- Baskets represent Liz’s income less expenditure on tea.
- She ________ with cash regardless of the amount of cash.

Parallel shifts do not change $x_T$.

$\phi^T(p_C^c, p_T, m_T) = \phi^T(p_C^c, p_T, m_T)$.

$\Delta x_T = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \Delta x^S_T$.

Income effect for tea is ______ for quasi-linear preferences.
An alternative way to compensate (adjust) income level for Slutsky decomposition.

- Slutsky: adjust $m$ to guarantee that $x^0$ is available under $p^2$.
- Hicks: adjust $m$ in the way that guarantees $u(x^0)$ under $p^2$.

When $\Delta p$ is small, the Slutsky substitution effect is almost the same as the Hicksian substitution effect.

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![Indifference Curves](image)

**Indifference Curves**

- $BL^0$ (Hicks)
- $BL^1$ (Slutsky)

---

**Introduction**

- Decomposing Effects
- Giffen is Income-Inferior
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**Now We Know**
- Decomposing total change in quantity demanded.
- Property of Giffen goods.
- Alternative compensation.

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