Chapter 5
Magnetostatics

Problem Set #5: 5.7, 5.9, 5.13 (Due Monday, April 8th)

5.1 Biot-Savart Law

So far we were concerned with static configuration of charges known as electrostatics. We now switch to magnetostatics where the charges are allowed to move but in a way that the created magnetic field is static corresponding to a steady state distribution of charges,

\[ \frac{\partial}{\partial t} \rho = 0. \]  

(5.1)

Consider a total charge \( Q \) inside some volume \( V \). Then the change of \( Q \) with time can be only due to the flow of charges through the boundary, i.e.

\[ \frac{\partial Q}{\partial t} = - \int J \cdot \hat{n} \, da, \]  

(5.2)

where \( J \) is the current density measured in units of charge per unit area per unit time. Using the divergence theorem we can rewrite (5.2) as

\[ \frac{\partial}{\partial t} \int \rho(x) d^3x = - \int \nabla \cdot J(x) d^3x \]  

(5.3)

which must be satisfied for an arbitrary volume element. Thus, we get a differential equation

\[ \frac{\partial}{\partial t} \rho + \nabla \cdot J = 0 \]  

(5.4)

known as the continuity equation, but in magnetostatics the equation is further simplified

\[ \nabla \cdot J = 0, \]  

(5.5)
due to the steady state condition (5.1).

It is an experimental fact that the steady states (5.1) produce an observable magnetic phenomena which depends on the current density \( \mathbf{J} \). Consider a wire of length \( \Delta l \), cross sectional area \( \Delta a \), pointing in \( \mathbf{l} \) and carrying an electric current \( I \), then at a distance \( \mathbf{r} \) from the wire there is a magnetic field which is
- directly proportional \( \Delta l \),
- directly proportional to \( I \),
- inversely proportional to the square of the distance \( r \) and
- points in the direction normal to the plane parallel to \( \mathbf{l} \) and \( \mathbf{r} \), i.e.

\[
\Delta \mathbf{B} = k \frac{I \Delta l}{r^2} \left( \mathbf{l} \times \mathbf{r} \right).
\] (5.6)

In a fixed coordinate system (and in SI units),

\[
\Delta \mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{J(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \Delta a \Delta l,
\] (5.7)

where the current density is

\[
\mathbf{J} = \frac{I}{\Delta a} \mathbf{l}.
\] (5.8)

To obtain an integral expression we replace \( \Delta a \Delta l \) with a volume integral over \( d^3x' \),

\[
\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{J(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d^3x'.
\] (5.9)

This is the Biot-Savart Law of magnetostatics (which is analogous to the Coulomb’s Law of electrostatics) discovered first by Oersted, and elaborated by Biot and Savart and later by Ampere.

Ampere’s experiment showed that the force on a current element \( d\mathbf{l} \) in a magnetic field is given by

\[
d\mathbf{F} = I (d\mathbf{l} \times \mathbf{B}).
\] (5.10)

This implies that the total force on a current density distribution is

\[
\mathbf{F} = \int \mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}) d^3x
\] (5.11)

and the total torque

\[
\mathbf{N} = \int \mathbf{x} \times (\mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x})) \, d^3x.
\] (5.12)
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For example, if the magnetic field \( B \) is generated by a closed current loop \#2 then the force which a closed current loop \#1 experiences can be calculated by substituting (5.9) into (5.10). Note that the volume integral is replaced by line integral since

\[ J = I \delta(x') \, dl. \]  

(5.13)

and the resulting force is

\[
F = \frac{\mu_0}{4\pi} I_1 I_2 \int \int \frac{dl_1 \times (dl_2 \times (l_1 - l_2))}{|l_1 - l_2|^3}
\]

\[
= \frac{\mu_0}{4\pi} I_1 I_2 \int \int \frac{-(dl_1 \cdot dl_2) (l_1 - l_2) + dl_2 (dl_1 \cdot (l_1 - l_2))}{|l_1 - l_2|^3}
\]

\[
= \frac{\mu_0}{4\pi} I_1 I_2 \int \int \frac{(dl_1 \cdot dl_2) (l_1 - l_2)}{|l_1 - l_2|^3} + \frac{\mu_0}{4\pi} I_1 I_2 \int \int dl_2 \left( dl_1 \cdot \nabla \frac{1}{|l_1 - l_2|} \right)
\]

(5.14)

(5.15)

the second term vanishes for closed loops, i.e.

\[
F = \frac{\mu_0}{4\pi} I_1 I_2 \int \int \frac{(dl_1 \cdot dl_2) (l_1 - l_2)}{|l_1 - l_2|^3}.
\]

Thus, the parallel currents attract and antiparallel current repel.

### 5.2 Ampere’s Law

Using

\[
\nabla \times \left( \frac{J(x')}{|x - x'|} \right) = -J(x') \times \nabla \frac{1}{|x - x'|} + \frac{1}{|x - x'|} \nabla \times J(x')
\]

\[
= -J(x') \times \nabla \frac{1}{|x - x'|}
\]

\[
= J(x') \times \frac{(x - x')}{|x - x'|^3}
\]

(5.16)

we can rewrite (5.9) as

\[
B(x) = \frac{\mu_0}{4\pi} \int J(x') \times \frac{(x - x')}{|x - x'|^3} d^3 x'
\]

\[
= \frac{\mu_0}{4\pi} \nabla \times \left( \int \frac{J(x')}{|x - x'|^3} d^3 x' \right).
\]

(5.17)
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Then it is convenient to define a vector potential

\[ \mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(x')}{|x - x'|} d^3x' \]  

such that

\[ \mathbf{B} = \nabla \times \mathbf{A} \]  

and, thus, the Gauss Law for magnetic fields (in the absence of magnetic charges) must be satisfied

\[ \nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0. \]  

As an aside, note that there is a freedom in defining \( \mathbf{A} \) which is called the gauge freedom or gauge symmetry. Indeed, if we redefine

\[ \mathbf{A}_{\text{new}}(x) = \mathbf{A}_{\text{old}}(x) + \nabla \psi(x), \]  

where \( \psi(x) \) is an arbitrary function, then the magnetic field would not change,

\[ \mathbf{B}_{\text{new}} = \nabla \times \mathbf{A}_{\text{new}} = \nabla \times (\mathbf{A}_{\text{old}} + \nabla \psi) = \nabla \times \mathbf{A}_{\text{old}} + \nabla \times \nabla \psi = \nabla \times \mathbf{A}_{\text{old}} = \mathbf{B}_{\text{old}}. \]  

So, we have a freedom of choosing \( \mathbf{A} \) which would make the calculations simple without effecting physically observable quantities such as \( \mathbf{B} \). This is similar to the freedom we had in choosing the electric potential since redefinitions

\[ \Phi_{\text{new}}(x) = \Phi_{\text{old}}(x) + C, \]  

would not change the physically observable electric field

\[ \mathbf{E}_{\text{new}} = -\nabla \Phi_{\text{new}} = -\nabla (\Phi_{\text{new}} + C) = -\nabla \Phi_{\text{old}} = \mathbf{E}_{\text{old}}. \]  

The new (gauge) transformation or new (gauge) symmetry is much bigger since it involves and arbitrary function \( \psi(x) \) and not just a constant \( C \). For example one can choose a Coulomb gauge such that

\[ \nabla \cdot \mathbf{A} = 0. \]  

(This is still not uniquely defined since we can add to the vector potential a gradient of an arbitrary scalar function which satisfies Laplace equation.) Of course to fully appreciate the gauge symmetry of electrodynamics we need to look at the quantum electrodynamics where the vector potential appears as a more fundamental quantity. For example in the double slit experiment a shift
in the interference pattern is observed if there is a non-vanishing magnetic flux in-between the two trajectories

$$\int \mathbf{B} \cdot \hat{n} da = \int (\nabla \times \mathbf{A}) \cdot \hat{n} da = \int \mathbf{A} \cdot dl. \quad (5.26)$$

This is known as the Aharonov-Bohm effect which disagrees with classical electrodynamics but can be explained using quantum mechanics where the same electron can simultaneously travel along two different trajectories. This shows that vector potential is more fundamental than magnetic field.

If we take a curl of (5.17) and make use of

$$\nabla \times \nabla \times \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \quad (5.27)$$

and

$$\nabla \frac{1}{|\mathbf{x} - \mathbf{x}'|} = -\nabla' \frac{1}{|\mathbf{x} - \mathbf{x}'|} \quad (5.28)$$

$$\nabla^2 \frac{1}{|\mathbf{x} - \mathbf{x}'|} = -4\pi \delta(\mathbf{x} - \mathbf{x}') \quad (5.29)$$

then we get

$$\nabla \times \mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \nabla \times \nabla \times \left( \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3 x' \right)$$

$$= \frac{\mu_0}{4\pi} \nabla \left( \int \nabla \cdot \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3 x' \right) - \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{x}') \nabla^2 \left( \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) d^3 x'$$

$$= -\frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{x}') \cdot \nabla \frac{1}{|\mathbf{x} - \mathbf{x}'|} d^3 x' + \mu_0 \mathbf{J}(\mathbf{x})$$

$$= -\frac{\mu_0}{4\pi} \int \nabla' \cdot \mathbf{J}(\mathbf{x}') \frac{1}{|\mathbf{x} - \mathbf{x}'|} d^3 x' + \mu_0 \mathbf{J}(\mathbf{x}). \quad (5.30)$$

In the case of magnetostatics (5.5) we get the Ampere’s law in differential form,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}(\mathbf{x}), \quad (5.31)$$

or in Coulomb gauge

$$\nabla \times \nabla \times \mathbf{A} = \mu_0 \mathbf{J} \quad (5.32)$$

$$\nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} \quad (5.33)$$

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \quad (5.34)$$
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and by integrating over some surface we obtain the Ampere’s law in integral form,

\[ \int \nabla \times \mathbf{B} \cdot \mathbf{n} \, da = \mu_0 \int \mathbf{J} \cdot \mathbf{n} \, da \]  \hspace{1cm} (5.35)

\[ \int \mathbf{B} \cdot d\mathbf{l} = \mu_0 I. \]  \hspace{1cm} (5.36)

Note that (5.31) implies that the magnetic field has a non-vanishing curl only if there is a current; and in a region with no currents the magnetic field is curl free and, thus, completely determined by the boundary conditions.

5.3 Magnetic multipole moments

Similarly to the multipole of expansion of the (scalar) electric potential, \( \Phi \), one can construct the multipole expansion of the vector (magnetic) potential, \( \mathbf{A} \). Consider the vector potential at \( \mathbf{x} \) due to a localized distribution of currents (near origin) parametrized by \( \mathbf{x}' \sim 0 \),

\[ \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d\mathbf{x}' \]  \hspace{1cm} (5.37)

where

\[ \mathbf{B} \equiv \nabla \times \mathbf{A} \]  \hspace{1cm} (5.38)

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \]  \hspace{1cm} (5.39)

If we expand

\[ \frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{|\mathbf{x}|} + \frac{\mathbf{x} \cdot \mathbf{x}'}{|\mathbf{x}|^3} + \ldots \]  \hspace{1cm} (5.40)

then

\[ \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{1}{|\mathbf{x}|} \int \mathbf{J}(\mathbf{x}') d\mathbf{x}' + \frac{\mu_0}{4\pi} \frac{1}{|\mathbf{x}|^3} \int (\mathbf{x} \cdot \mathbf{x}') \mathbf{J}(\mathbf{x}') d\mathbf{x}' + \ldots \]  \hspace{1cm} (5.41)

For an arbitrary pair of function \( f(\mathbf{x}) \) and \( g(\mathbf{x}) \) and localized \( \mathbf{J}(\mathbf{x}) \) we have

\[ \int g \mathbf{J} \cdot \nabla' f d^3x' = -\int f \nabla' \cdot (g \mathbf{J}) f d^3x' + \int f g \mathbf{J} \cdot \mathbf{n} \, da \]  \hspace{1cm} (5.42)

\[ \int g \mathbf{J} \cdot \nabla' f d^3x' = -\int f g \nabla' \cdot \mathbf{J} f d^3x' - \int f \mathbf{J} \nabla' g d^3x' \]  \hspace{1cm} (5.43)
or
\[ \int g \mathbf{J} \cdot \mathbf{\nabla'} f \, d^3 x' + \int f g \mathbf{\nabla'} \cdot \mathbf{J} f \, d^3 x' + \int f \mathbf{J} \mathbf{\nabla'} g \, d^3 x' = 0. \] (5.44)

By substituting in (5.44), \( f(x') = 1 \) and \( g(x') = x'_i \) we get
\[ 0 + \int x'_i \mathbf{\nabla'} \cdot \mathbf{J} f \, d^3 x' + \int J_i d^3 x' = 0, \] (5.45)
\[ \int J_i d^3 x' = - \int x'_i \mathbf{\nabla'} \cdot \mathbf{J} f \, d^3 x' \] (5.46)

and \( f(x) = x'_i \) and \( g(x) = x'_j \) we get
\[ \int x'_i J_j d^3 x' + \int x'_i x'_j \mathbf{\nabla'} \cdot \mathbf{J} f \, d^3 x' + \int x'_j J_i d^3 x' = 0 \] (5.47)
\[ \int x'_i J_j d^3 x' + \int x'_j J_i d^3 x' = - \int x'_i x'_j \mathbf{\nabla'} \cdot \mathbf{J} f \, d^3 x' \] (5.48)

However for the steady states of magnetostatics the current is divergence-less (i.e. \( \nabla \cdot \mathbf{J} = 0 \)) and (5.46), (5.48) are simplified
\[ \int J_i d^3 x' = 0 \] (5.49)
\[ \int (x'_i J_j + x'_j J_i) d^3 x' = 0 \] (5.50)

From (5.49) the magnetic moment of a localized current density must vanish and (5.41) becomes
\[ \mathbf{A}(x) = \mu_0 \frac{1}{4\pi |x|^3} \int (x \cdot x') \mathbf{J}(x') \, dx' + \ldots \] (5.51)
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Moreover, (5.50) can be used to rewrite (5.51) as

\[
\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{1}{|\mathbf{x}|^3} \int d\mathbf{x}' \sum_{i,j=1,2,3} x_j x'_j J_i \hat{x}_i
\]

\[
= -\frac{1}{2} \frac{\mu_0}{4\pi} \frac{1}{|\mathbf{x}|^3} \int d\mathbf{x}' \sum_{i,j=1,2,3} x_j (x'_i J_j - x'_j J_i) \hat{x}_i
\]

\[
= -\frac{1}{2} \frac{\mu_0}{4\pi} \frac{1}{|\mathbf{x}|^3} \int d\mathbf{x}' \hat{x}_1 (x_2 (x'_1 J_2 - x'_2 J_1) - x_3 (x'_3 J_3 - x'_3 J_3))
\]

\[
- \frac{1}{2} \frac{\mu_0}{4\pi} \frac{1}{|\mathbf{x}|^3} \int d\mathbf{x}' \hat{x}_2 (x_3 (x'_2 J_3 - x'_3 J_2) - x_1 (x'_1 J_2 - x'_2 J_1))
\]

\[
- \frac{1}{2} \frac{\mu_0}{4\pi} \frac{1}{|\mathbf{x}|^3} \int d\mathbf{x}' \hat{x}_3 (x_1 (x'_3 J_1 - x'_3 J_3) - x_2 (x'_2 J_3 - x'_2 J_3))
\]

\[
= -\frac{1}{2} \frac{\mu_0}{4\pi} \frac{1}{|\mathbf{x}|^3} \mathbf{x} \times (\int d\mathbf{x}' \mathbf{x} \times (x'_2 J_3 - x'_3 J_2) \hat{x}_1 + (x'_3 J_1 - x'_1 J_3) \hat{x}_2 + (x'_1 J_2 - x'_2 J_1) \hat{x}_3)
\]

\[
= -\frac{1}{2} \frac{\mu_0}{4\pi} \frac{1}{|\mathbf{x}|^3} \mathbf{x} \times (\int d\mathbf{x}' \mathbf{x} \times \mathbf{J}(\mathbf{x}')d\mathbf{x'}). 
\]

(5.52)

The same calculation could have been done using the so-called Levi-Civita (or completely antisymmetric tensor) defined as

\[
\epsilon_{ijk} = \begin{cases} 
+1 & \text{if } (i, j, k) \text{ is } (1, 2, 3), (3, 1, 2) \text{ or } (2, 3, 1) \\
-1 & \text{if } (i, j, k) \text{ is } (1, 3, 2), (2, 1, 3) \text{ or } (3, 2, 1) \\
0 & \text{if } i = j, j = k, \text{ or } i = k. 
\end{cases} 
\]

(5.53)

For example, the cross product of three-vectors can be expressed as

\[
\mathbf{A} \times \mathbf{B} = \sum_{i,j,k=1,2,3} \epsilon_{ijk} \hat{x}_i A_j B_k, 
\]

(5.54)
or using Einstein summation notation (i.e. always sum over repeated indices) as

\[
\mathbf{A} \times \mathbf{B} = \epsilon_{ijk} \hat{x}_i A_j B_k. 
\]

(5.55)

For instance

\[
(A \times B)_k = \epsilon_{ijk} A_j B_j 
\]

(5.56)
or

\[
(A \times B)_1 = A_2 B_3 - A_3 B_2 
\]

(5.57)
\[
(A \times B)_2 = A_3 B_1 - A_1 B_3 
\]

(5.58)
\[
(A \times B)_3 = A_1 B_2 - A_2 B_1. 
\]

(5.59)
Then

\[ A(x) = \frac{1}{2} \mu_0 \frac{1}{24\pi} \int dx' \sum_{i,j} x_j (x'_i J_j - x'_j J_i) \hat{x}_i \]

\[ = -\frac{1}{2} \mu_0 \frac{1}{24\pi} \int dx' \sum_{i,j} \epsilon_{ijk} x_i x_j (x' \times J)_k \]

\[ = -\frac{1}{2} \mu_0 \frac{1}{24\pi} x \times (\int x' \times J(x') dx'). \quad (5.60) \]

If we now define a magnetic dipole moment as

\[ m \equiv \frac{1}{2} \int x' \times J(x') dx' \quad (5.61) \]

then (5.52) takes the following form

\[ A(x) = \frac{\mu_0 m \times x}{4\pi |x|^3}. \quad (5.62) \]

which is similar to the electric potential produced by the electric dipole moment (4.10),

\[ \Phi(x) = \frac{1}{4\pi \epsilon_0} \frac{p \cdot x}{|x|^3}. \quad (5.63) \]

The magnetic dipole can also be expressed through the magnetic dipole moment density

\[ m = \int M(x') dx'. \quad (5.64) \]

defined as

\[ M(x) \equiv \frac{1}{2} x' \times J(x'). \quad (5.65) \]

Then the magnetic field is given by

\[ B = \nabla \times A \]

\[ = \frac{\mu_0}{4\pi} \nabla \times \frac{m \times x}{|x|^3} \]

\[ = \frac{\mu_0}{4\pi} \frac{3\hat{r}(\hat{r} \cdot m) - m}{r^3} \quad (5.66) \]

to be compared with the electric field produced due to the electric dipole moment (4.14),

\[ E = \frac{1}{4\pi \epsilon_0} \frac{3\hat{r}(\hat{r} \cdot p) - p}{r^3}. \quad (5.67) \]
For example, if the current $I$ flows in an arbitrary closed loop of area $A$ confined to the $z = 0$ plane then the magnetic moment point along $z$ axis with magnitude determined from (5.61), i.e.

$$|\mathbf{m}| = \frac{1}{2} \int \mathbf{x} \times d\mathbf{l} = IA. \quad (5.68)$$

### 5.4 External magnetic field

Consider a localized current density from the previous section in the presence of external magnetic field, $\mathbf{B}(\mathbf{x})$. According to Ampere’s law the distribution would experience a force (5.11),

$$\mathbf{F} = \int \mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}) d^{3}x \quad (5.69)$$

and the resulting torque (5.12),

$$\mathbf{N} = \int \mathbf{x} \times (\mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x})) d^{3}x. \quad (5.70)$$

If the external magnetic field is a slowly varying function (near appropriately chosen origin), then one can expand it (around the origin) to the linear order,

$$\mathbf{B}(\mathbf{x}) \approx \mathbf{B}(0) + (\mathbf{x} \cdot \nabla) \mathbf{B}(0). \quad (5.71)$$

Then the force (5.69) is given by

$$\mathbf{F} = \int d^{3}x \mathbf{J}(\mathbf{x}) \times (\mathbf{B}(0) + (\mathbf{x} \cdot \nabla) \mathbf{B}(0))$$

$$= \epsilon_{ijk} \mathbf{\hat{x}}_i \left( \int d^{3}x' J_j(\mathbf{x}') \right) B_k(0) + \epsilon_{ijk} \mathbf{\hat{x}}_i \int d^{3}x' J_j(\mathbf{x}') \mathbf{x}' \cdot \nabla B_k(0) \quad (5.72)$$

For a localized distribution of currents the first (or zeroth order) term vanishes and the leading contribution comes from the second (or linear order term)

$$F_i = \epsilon_{ijk} \left( \int d^{3}x' J_j(\mathbf{x}') x'_{l} \right) \frac{\partial}{\partial x_l} B_k$$

$$= \frac{1}{2} \epsilon_{ijk} \int d^{3}x' \left( J_j(\mathbf{x}') x'_{l} - J_l(\mathbf{x}') x'_{j} \right) \frac{\partial}{\partial x_l} B_k(0) + \frac{1}{2} \epsilon_{ijk} \int d^{3}x' (\mathbf{J}(\mathbf{x}') \times \mathbf{x}')_{l} \frac{\partial}{\partial x_l} B_k(0)$$

$$= \epsilon_{ijk} m_k \frac{\partial}{\partial x_l} B_k(0)$$

$$= \epsilon_{ijk} (\mathbf{m} \times \nabla)_j B_k(0) \quad (5.73)$$
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and thus,
\[ F = (m \times \nabla) \times B(0) \]
\[ = \nabla (m \cdot B(0)) - m (\nabla \cdot B(0)) \]
\[ = \nabla (m \cdot B(0)) \]  
(5.74)
since the magnetic field is divergence-less (i.e. \( \nabla \cdot B = 0 \)).

Expression for torque is found by inserting the zeroth order term of (5.71) into (5.70),
\[ N = \int x' \times (J(x') \times B(0)) \, d^3x' \]
\[ = \int (x' \cdot B(0)) J(x') d^3x' - \int (x' \cdot J(x')) B(0) d^3x'. \]  
(5.75)
The second integral vanishes for a localized distribution of currents and the first integral is the same as the second integral in (5.72) with \( \nabla B_k(0) \) replaced with \( B(0) \). The final expression is
\[ N = m \times B(0). \]  
(5.76)
If we define the potential energy, \( U \), as a quantity whose negative gradient is force
\[ F = -\nabla U \]  
(5.77)
then according to (5.74) the potential energy (up to an additive constant) is given by
\[ U = -m \cdot B. \]  
(5.78)
Clearly the energy is minimized when the magnetic moment \( m \) is in the direction of the magnetic field \( B \).

5.5 Magnetostatics in Ponderable media

When electric currents a placed in a material the material might respond (or magnetize) to the generated magnetic fields analogous to how the materials respond to electric fields. The magnetization of the material can be induced or can be permanent as in permanent magnets. To obtain a macroscopic description of such systems one must average (or coarse-grain) over macroscopically small, but microscopically large regions. The averaging is particularly simple for steady state currents of magnetostatics
\[ \nabla \times B_{\text{micro}} = 0 \quad \Rightarrow \quad \nabla \times B_{\text{macro}} = 0. \]  
(5.79)
This means that we can use a concept of coarse-grained vector potential defined as before

\[ \nabla \times \mathbf{A}_{\text{macro}} \equiv \mathbf{B}_{\text{macro}}. \tag{5.80} \]

The molecules in ponderable media can be considered as small loops of current with non-vanishing magnetic moment which do not move on macroscopic distances. However, according to (5.78) they should all rotate in the direction of the external magnetic field to minimize the potential energy. The overall effect of the induced magnetic dipole moments is defined macroscopically as a sum over averaged dipole moment density,

\[ \mathbf{M}(\mathbf{x}) = \sum_i N_i \langle \mathbf{m}_i \rangle, \tag{5.81} \]

where \( \langle \mathbf{m}_i \rangle \) is the magnetic dipole moment and \( N_i \) is the number density of the \( i \)th type of molecule.

Consider a small volume \( \Delta V \) of the ponderable media. Then the vector potential at some distant point \( \mathbf{x} \) can be expanded into multipole moments as

\[ \Delta \mathbf{A} = \frac{\mu_0}{4\pi} \left[ \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \Delta V + \frac{\mathbf{M}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \Delta V \right] \tag{5.82} \]

If we treat \( \Delta V \) as an infinitesimal \( d^3x' \) then

\[ \mathbf{A}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \left[ \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \frac{\mathbf{M}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \right] \]
\[ = \frac{1}{4\pi\epsilon_0} \int d^3x' \left[ \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \mathbf{M}(\mathbf{x}') \times \nabla' \left( \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right] \]
\[ = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} \left[ \mathbf{J}(\mathbf{x}') + \nabla' \times \mathbf{M}(\mathbf{x}') \right]. \tag{5.83} \]

It is convenient to separate the macroscopic magnetic field into two parts

\[ \frac{\mathbf{B}(\mathbf{x})}{\mu_0} = \mathbf{H}(\mathbf{x}) + \mathbf{M}(\mathbf{x}). \tag{5.84} \]

with contributions from the excess (or applied) currents

\[ \nabla \times \mathbf{H} = \mathbf{J} \tag{5.85} \]

and the magnetization (or induced) currents

\[ \nabla \times \mathbf{M} = \mathbf{J}_m. \tag{5.86} \]
Then the combined effect is given by

\[ \nabla \times \left( \frac{B}{\mu_0} \right) = \nabla \times H + \nabla \times M = J + J_m. \] (5.87)

Note a similar role played by \( P, D \) and \( E \) in electrostatic described by

\[ \begin{align*}
\nabla \cdot P &= -\rho_p \quad (5.88) \\
\nabla \cdot D &= \rho \quad (5.89) \\
\nabla \times E &= 0 \quad (5.90)
\end{align*} \]

to the role played by \( M, H \) and \( B \) in magnetostatics

\[ \begin{align*}
\nabla \times M &= J_M \quad (5.91) \\
\nabla \times H &= J \quad (5.92) \\
\n\nabla \cdot B &= 0 \quad (5.93)
\end{align*} \]

The simplest types of materials are diamagnetic (i.e. \( \mu < \mu_0 \)) and paramagnetic (i.e. \( \mu > \mu_0 \)) whose response to the applied field is linear and isotropic, i.e.

\[ B = \mu H \quad (5.94) \]

where \( \mu \) is the magnetic permeability typically very close to \( \mu_0 \). If the material is homogeneous (uniform) then all of the solutions obtained in the vacua are valid with a trivial replacement \( \mu \rightarrow \mu_0 \). If the medium, for example, consists of two regions with different magnetic properties, then we can derive boundary conditions that \( H \) and \( B \) must satisfy. An integral over a closed contour enclosing the boundary gives

\[ \int \mathbf{H} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{H}) \cdot \hat{n} d\alpha = \int \mathbf{J} \cdot \mathbf{n} d\alpha \quad (5.95) \]

and

\[ \hat{n} \times (H_2 - H_1) = K \quad (5.96) \]

or

\[ \int \mathbf{M} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{M}) \cdot \hat{n} d\alpha = \int J_M \cdot \mathbf{n} d\alpha \quad (5.97) \]

and

\[ \hat{n} \times (M_2 - M_1) = K_m \quad (5.98) \]

where \( K \) and \( K_m \) are the surface current densities of the applied and induced magnetization currents respectively. Moreover, a volume integral enclosing
the boundary leads to
\[
\int \nabla \cdot \mathbf{B} \, d^3x = 0 \tag{5.99}
\]
\[
\int \mathbf{B} \cdot \hat{n} \, da = 0 \tag{5.100}
\]
\[
\hat{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0. \tag{5.101}
\]

5.6 Linear media

For linear (i.e. \( \mathbf{B} = \mu \mathbf{H} \)) and homogeneous (i.e. \( \mu(\mathbf{x}) = \text{const} \)) media we can rewrite (5.92) in terms of the vector potential as,

\[
\nabla \times \left( \frac{1}{\mu} \nabla \times \mathbf{A} \right) = \mathbf{J} \tag{5.102}
\]
\[
\nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J}. \tag{5.103}
\]

Then it is convenient to work in Coulomb gauge (i.e. \( \nabla \cdot \mathbf{A} = 0 \)) so that (5.103) reduces to the form of (5.34) with \( \mu_0 \) replaced with \( \mu \),

\[
\nabla^2 \mathbf{A} = -\mu \mathbf{J}. \tag{5.104}
\]

When the current density vanishes (i.e. \( \mathbf{J} = 0 \)) the solution can be obtained using a magnetic scalar potential (similarly to the electric potential),

\[
\mathbf{H} = -\nabla \Phi_M. \tag{5.105}
\]

and (5.93) becomes

\[
\nabla \cdot (\mu \nabla \Phi_M) = 0 \tag{5.106}
\]
\[
\nabla^2 \Phi_M = 0 \tag{5.107}
\]

in each homogeneous region (i.e. \( \mu(\mathbf{x}) = \text{const} \)). Alternatively, one can choose a scalar potential \( \Psi_M = \mu \Phi_M \).
For example, let us calculate the effect of an unmagnitized sphere of radius \(a\) made of linear magnetic material placed in an initially uniform magnetic field \(\mathbf{B} = B_0 \hat{z}\). Since there are no applied currents we can solve a Laplace equation for the scalar potential

\[
\nabla^2 \Psi_M = 0
\]

(5.108)

to determine

\[
\mathbf{B} = -\nabla \Psi_M.
\]

(5.109)

Because of the azimuthal symmetry we can expand the solution in Legendre polynomials

\[
\Psi_M(r, \theta, \phi) = \sum_l \left( C_l r^l + D_l r^{-l-1} \right) P_l(\cos \theta).
\]

(5.110)

Outside of the sphere and at very large \(r\) the magnetic field should be \(\mathbf{B} = B_0 \hat{z}\) or

\[
\Psi_{\text{out}}^M \approx -B_0 r \cos \theta
\]

(5.111)

and thus,

\[
C_l = \begin{cases} 
0 & \text{for } i = 0 \\
-B_0 & \text{for } i = 1 \\
0 & \text{for } i \geq 2
\end{cases}
\]

(5.112)

or

\[
\Psi_{\text{out}}^M = -B_0 r \cos \theta + \sum_l D_l r^{-l-1} P_l(\cos \theta).
\]

(5.113)

Inside the sphere a finiteness of the Magnetic field implies that

\[
\Psi_{\text{in}}^M = \sum_l C_l r^l P_l(\cos \theta).
\]

(5.114)

To determine the constants \(D_l\)’s and \(C_l\)’s we must match the solutions at the boundary using (5.101)

\[
\hat{r} \cdot (\nabla \Psi_{\text{out}}^M - \nabla \Psi_{\text{in}}^M) = 0
\]

\[
\left[ \frac{\partial \Psi_{\text{out}}^M}{\partial r} \right]_{r=a} = \left[ \frac{\partial \Psi_{\text{in}}^M}{\partial r} \right]_{r=a}
\]

\[
-B_0 \cos \theta - \sum_{l=0}^\infty (l + 1) D_l a^{-l-2} P_l(\cos \theta) = \sum_{l=1}^\infty l C_l a^{l-1} P_l(\cos \theta).
\]

(5.115)
and (5.96)

\[ \hat{r} \times \left( \frac{1}{\mu_0} \nabla \Psi_M^{\text{out}} - \frac{1}{\mu} \nabla \Psi_M^{\text{in}} \right) = 0 \]

\[ \frac{1}{\mu_0} \left[ \frac{\partial \Psi_M^{\text{out}}}{\partial \theta} \right]_{r=a} = \frac{1}{\mu} \left[ \frac{\partial \Psi_M^{\text{in}}}{\partial \theta} \right]_{r=a} \]

\[ \frac{1}{\mu_0} \left( -B_0 a \frac{d}{d\theta} P_1(\cos \theta) + \sum_{l=1} D_l a^{-l-1} \frac{d}{d\theta} P_l(\cos \theta) \right) = \frac{1}{\mu} \sum_{l=1} C_l a^l \frac{d}{d\theta} P_l(\cos \theta) \]

By equating the respective coefficients we get

\[ D_0 = 0 \]

\[ B_0 - 2D_1 a^{-3} = C_1 \]

\[-(l + 1) D_l a^{-l-2} = lC_l a^{-l-1} \text{ for } l > 1 \]

and

\[ \frac{1}{\mu_0} (-B_0 a + D_1 a^{-2}) = \frac{1}{\mu} C_1 a \]

\[ \frac{1}{\mu_0} D_1 a^{-l-1} = \frac{1}{\mu} C_l a^l \text{ for } l > 1. \]

which are satisfied only when

\[ D_1 = C_l = 0 \text{ for } l > 1 \]

\[ C_1 = \frac{-3\mu}{\mu + 2\mu_0} B_0 \]

\[ D_1 = \frac{\mu - \mu_0}{\mu + 2\mu_0} a^3 B_0. \]

By putting everything together we obtain our final solution

\[ \Psi_M^{\text{out}} = -B_0 r \cos \theta + \frac{\mu - \mu_0}{\mu + 2\mu_0} a^3 B_0 r^{-2} \cos \theta \]

\[ \Psi_M^{\text{in}} = -\frac{3\mu}{\mu + 2\mu_0} B_0 r \cos \theta \]

which should be compared with (4.68) and (4.69) obtained by solving a completely analogous electrostatic problem. The final expression for the magnetic field is then

\[ \mathbf{B}^{\text{out}} = -\left( \hat{\mathbf{r}} \frac{\partial}{\partial r} \Psi_M^{\text{out}} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \Psi_M^{\text{out}} \right) = B_0 \hat{\mathbf{z}} + B_0 \frac{\mu - \mu_0}{\mu + 2\mu_0} \left( \frac{a}{r} \right)^3 (3 \cos \theta \hat{\mathbf{r}} - \hat{\mathbf{z}}) \]

\[ \mathbf{B}^{\text{in}} = -\left( \hat{\mathbf{r}} \frac{\partial}{\partial r} \Psi_M^{\text{in}} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \Psi_M^{\text{in}} \right) = \frac{3\mu}{\mu + \mu_0} B_0 \hat{\mathbf{z}}. \]
5.7 Hard ferromagnets

Another interesting example involves the so-called “hard” ferromagnets whose magnetization is nearly independent of the applied magnetic field. In the case of vanishing currents (5.93) is written using (5.107) as

\[ \nabla \cdot \mathbf{B} = 0 \]  
\[ \mu_0 \nabla \cdot (\mathbf{H} + \mathbf{M}) = 0 \]  
\[ \nabla^2 \Phi_M = -\nabla \cdot \mathbf{M} \]  
\[ \nabla^2 \Phi_M = -\rho_M \]  

where

\[ \rho_M \equiv -\nabla \cdot \mathbf{M} \]

is called the effective magnetic charge density. Clearly, the solution for the magnetic scalar potential (just like for the electric scalar potential) must be given by

\[ \Phi_M(\mathbf{x}) = -\frac{1}{4\pi} \int \frac{\nabla' \cdot \mathbf{M}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' \]  

(5.121)

For a localized distribution of \( \mathbf{M}(\mathbf{x}') \) one can integrate by parts and ignore the boundary term to get

\[ \Phi_M(\mathbf{x}) = -\frac{1}{4\pi} \int \mathbf{M}(\mathbf{x}') \cdot \nabla' \frac{1}{|\mathbf{x} - \mathbf{x}'|} d^3x' \]
\[ = \frac{1}{4\pi} \int \mathbf{M}(\mathbf{x}') \cdot \nabla \frac{1}{|\mathbf{x} - \mathbf{x}'|} d^3x' \]
\[ = \frac{1}{4\pi} \nabla \int \frac{\mathbf{M}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x'. \]  

(5.122)

Note that far from the magnetization region

\[ \Phi_M(\mathbf{x}) \approx -\frac{1}{4\pi} \nabla \left( \frac{1}{|\mathbf{x}|} \right) \cdot \int \mathbf{M}(\mathbf{x}') d^3x' = \frac{\mathbf{m} \cdot \mathbf{x}}{4\pi \epsilon_0^2}. \]  

(5.123)

If the “hard” ferromagnets have a finite size with a boundary beyond which the magnetization vanishes, then the divergence theorem can be used to calculate the effective magnetic charge density

\[ \sigma_M = \mathbf{n} \cdot \mathbf{M} \]  

(5.124)

and (5.121) generalizes to

\[ \Phi_M(\mathbf{x}) = -\frac{1}{4\pi} \int \frac{\nabla' \cdot \mathbf{M}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' + \frac{1}{4\pi} \int \frac{\mathbf{n}' \cdot \mathbf{M}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} da'. \]  

(5.125)
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Note that for a homogeneous magnetization the first term vanishes.
For the problem of “hard” ferromagnets with currents it is convenient to work with vector potential in Coulomb gauge (i.e. $\nabla \cdot A = 0$)

\[
\begin{align*}
\nabla \times H &= 0 \\
\nabla \times \left( \frac{B}{\mu_0} - M \right) &= 0 \\
\nabla^2 A &= -\mu_0 \nabla \times M \\
\nabla^2 A &= -\mu_0 J_M
\end{align*}
\]

(5.126) \hspace{1cm} (5.127) \hspace{1cm} (5.128) \hspace{1cm} (5.129)

where $J_M$ is the magnetization current (5.86). In the absence of boundary the solution was already derived in (5.83),

\[
A(x) = \frac{\mu_0}{4\pi} \int \frac{\nabla' \times M(x')}{|x - x'|} d^3x'
\]

(5.130)

and a more general solution

\[
A(x) = \frac{\mu_0}{4\pi} \int \frac{\nabla' \times M(x')}{|x - x'|} d^3x' - \frac{1}{4\pi} \int \hat{n}' \times M(x') |x - x'| da'.
\]

(5.131)

can be obtained by including the boundary term in the differentiation by parts in (5.83).

5.8 Faraday’s Law of Induction

The first observation of the effect of the time-dependent magnetic fields were made by Faraday who observed that the change in time of the total magnetic flux through some surface creates an electromotive force (and therefore electric current) on the loop bounding the surface, i.e.

\[
E = \int E' \cdot dl = -k \frac{d}{dt} \int B \cdot \hat{n} da
\]

(5.132)

where $E'$ is the electric field at $dl$ in the coordinate system where $dl$ is at rest. To determine $k$ we can make an assumption of Galilean invariance (i.e. physical laws are invariant under Galilean transformation $(x, t) \rightarrow (x' = x - vt, t' = t)$).

Consider a loop moving with some velocity $v$ (whose spatial derivatives
vanish), then
\[
\frac{d}{dt} \int \mathbf{B} \cdot \hat{n} \, da = \int \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{n} \, da + \int (\mathbf{v} \cdot \nabla) \mathbf{B} \, da
\]
\[
= \int \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{n} \, da + \int \nabla \times (\mathbf{B} \times \mathbf{v}) \, da + \int \mathbf{v} \cdot \nabla \times \mathbf{B} \, da
\]
\[
= \int \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{n} \, da + \int (\mathbf{B} \times \mathbf{v}) \cdot d\mathbf{l} \quad (5.133)
\]
By inserting (5.133) into (5.132) we obtain
\[
\int (\mathbf{E}' - k (\mathbf{v} \times \mathbf{B})) \cdot d\mathbf{l} = -k \int \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{n} \, da. \quad (5.134)
\]
Alternatively one can consider the same loop but in a coordinate system moving with the loop and then \( \mathbf{v} = 0 \) and
\[
\int \mathbf{E} \cdot d\mathbf{l} = -k \int \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{n} \, da \quad (5.135)
\]
where \( \mathbf{E} \) is the electric field in the new frame. The Galilean invariance implies that
\[
\mathbf{E} = \mathbf{E}' - k (\mathbf{v} \times \mathbf{B}) \quad (5.136)
\]
or
\[
\mathbf{E}' = \mathbf{E} + k (\mathbf{v} \times \mathbf{B}). \quad (5.137)
\]
A test charge at rest experiences an electric force
\[
\mathbf{F}' = q \mathbf{E}' \quad (5.138)
\]
but in a moving reference frame the same charge represents a current \( \mathbf{J} = q \mathbf{v} \delta(x - x_0) \) and experiences an electric as well as magnetic forces,
\[
\mathbf{F} = q \mathbf{E} + \int \mathbf{J}(x) \times \mathbf{B}(x) \, d^3x
\]
\[
= q (\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (5.139)
\]
Therefore the Galilean invariance implies that \( k = 1 \) and the Faraday’s Law takes the following form
\[
\int \mathbf{E}' \cdot d\mathbf{l} = -\frac{d}{dt} \int \mathbf{B} \cdot \hat{n} \, da. \quad (5.140)
\]
If we choose the reference frame where the current is held fixed
\[
\int \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial}{\partial t} \mathbf{B} \cdot \hat{n} \, da \quad (5.141)
\]
then
\[ \int \left( \nabla \times \mathbf{E} + \frac{\partial}{\partial t} \mathbf{B} \right) \cdot \mathbf{n} da = 0 \] (5.142)
must be satisfied for an arbitrary integration volume and we obtain a differential form of the Faraday’s law
\[ \nabla \times \mathbf{E} + \frac{\partial}{\partial t} \mathbf{B} = 0 \] (5.143)
which is a time-dependent generalization of the electrostatic equation \( \nabla \times \mathbf{E} = 0 \).

A changing magnetic flux creates an electromotive force and thus does work on a current per unit time (or power)
\[ \frac{dW}{dt} = -I \mathcal{E} = I \frac{d}{dt} \int \mathbf{B} \cdot \mathbf{n} da. \] (5.144)
For linear magnetic material one can show that
\[ W = \frac{1}{2} \int \mathbf{H} \cdot \mathbf{B} d^3x \] (5.145)
and the magnetic potential energy can be defined as
\[ w = \frac{1}{2\mu} \mathbf{B}^2. \] (5.146)