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Limit distributions for products of sums

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Abstract

Let $\{X, X_n,\ n \geqslant 1\}$ be a sequence of independent and identically distributed positive random variables and set $S_n = \sum_{j=1}^n X_j$ for $n \geqslant 1$. This paper proves that properly normalized products of the partial sums, $(\prod_{j=1}^n S_j/n!\mu^n)^{\mu/A_n}$, converges in distribution to some nondegenerate distribution when X is in the domain of attraction of a stable law with index $\alpha \in (1, 2]$.

MSC: 60F05

Keywords: Stable laws; Product of sums; Limit distribution

1. Introduction

Let $\{X, X_n,\ n \geqslant 1\}$ be a sequence of independent and identically distributed random variables and define the partial sum $S_n = \sum_{j=1}^n X_j$ for $n \geqslant 1$. In the past century, the partial sum S_n has been the most popular topic for study. Such well-known classic laws as the distributional laws, strong laws of large numbers, and the law of the iterated logarithm all describe the partial sum.

In this paper we assume that $X \geqslant 0$. In a recent paper by Rempała and Wesołowski (2002) it is showed under the assumption $E(X^2) < \infty$ that

$$\left(\frac{\prod_{j=1}^n S_j}{n!\mu^n}\right)^{\frac{1}{\gamma\sqrt{n}}} \xrightarrow{d} \mathrm{e}^{\sqrt{2}\mathcal{N}}, \tag{1.1}$$

where $\mathcal{N}$ is a standard normal random variable, $\mu = E(X)$ and $\gamma = \sigma/\mu$ with $\sigma^2 = \mathrm{Var}(X)$. Obviously (1.1) provides an alternative method for the inference of μ. The study of (1.1) was motivated by a study by Arnold and Villaseñor (1998) who considered the limit distributions for sums of records.

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Arnold and Villaseñor (1998) focused on a special case when X is a unit exponential random variable. In this case,

$$\left(\prod_{j=1}^{n}\frac{S_j}{j}\right)^{1/\sqrt{n}} \xrightarrow{d} e^{\sqrt{2}\mathcal{N}}. \tag{1.2}$$

The original result in Arnold and Villaseñor (1998) was stated in an equivalent form with a logarithm taken on both sides of (1.2):

$$\frac{\sum_{j=1}^{n}\log S_j - n\log n + n}{\sqrt{2n}} \xrightarrow{d} \mathcal{N}. \tag{1.3}$$

This shows that the asymptotic behavior for the products of sums of positive random variables resembles that for sums of independent random variables under certain circumstances.

The present paper focuses on the study of the limit distributions for products of sums of positive random variables in more general settings. Precisely speaking, we will assume that the positive random variable X is in the domain of attraction of a stable law with index $\alpha \in (1,2]$. The main result and its proof will be given in the next section.

2. Main theorem

As in the introduction, we let $\{X, X_n,\ n \geqslant 1\}$ be a sequence of independent and identically distributed positive random variables and set $S_n = \sum_{j=1}^{n} X_j$ for $n \geqslant 1$ and assume X is in the domain of attraction of a stable law with index $\alpha \in (1,2]$. Note that $E(X) < \infty$ when X is the domain of attraction of a stable law with an index $\alpha \in (1,2]$.

Recall that a sequence of independent and identically distributed random variables $\{X, X_n,\ n \geqslant 1\}$ is said to be in the domain of attraction of a stable law $\mathscr{L}$ if there exist constants $A_n > 0$ and $B_n \in R$ such that

$$\frac{S_n - B_n}{A_n} \xrightarrow{d} \mathscr{L}, \tag{2.1}$$

where $\mathscr{L}$ is one of the stable distributions with index $\alpha \in (0,2]$.

The following theorem is well known (see e.g., Hall, 1981 or Bingham et al., 1987).

Theorem 2.1 (Stability Theorem). *The general stable law is given, to within type, by a characteristic function of one of the following forms*:

(i) $\phi(t) = \exp\{-t^2/2\}$ (*normal case*, $\alpha = 2$);
(ii) $\phi(t) = \exp\{-|t|^{\alpha}(1 - i\beta(\operatorname{sgn} t)\tan\frac{1}{2}\pi\alpha\}$ $(0 < \alpha < 1$ *or* $1 < \alpha < 2)$, $-1 \leqslant \beta \leqslant 1$;
(iii) $\phi(t) = \exp\{-|t|(1 + i\beta(\operatorname{sgn} t)2/\pi \log|t|\}$ $(\alpha = 1,\ -1 \leqslant \beta \leqslant 1)$.

It is worth mentioning that in Theorem 2.1, β is the skewness parameter. In our paper, $\beta = 1$ since X is a non-negative random variable.

Let F denote the distribution function of $X' = X - \mu$. Define the generalized inverse of $1/(1-F)$ by

$$U(x) = \inf\left\{t\colon \frac{1}{1-F(t)} \geqslant x\right\}.$$

Write

$$S(x) = E[(X')^2]I(|X'| \leqslant x), \quad \text{for } x > 0,$$

and denote the generalized inverse of $x^2/S(x)$ by $V(x)$

$$V(x) = \inf\left\{t\colon \frac{t^2}{S(t)} \geqslant x\right\}.$$

One can always take $B_n = n\mu$ in our setting. Throughout this paper we will take $A_n = U(n)$ if $\alpha < 2$, and $A_n = V(n)$ if $\alpha = 2$. Then from Loève (1977), the limit $\mathscr{L}$ in (2.1) has a characteristic function as in Theorem 2.1.

Theorem 2.2. *Assume that the non-negative random variable X is in the domain of attraction of a stable law with index $\alpha \in (1,2]$ with $\mu = E(X)$. The constants A_n are defined as above so that the limit $\mathscr{L}$ in* (2.1) *has a characteristic function as in Theorem* 2.1. *Then*

$$\left(\frac{\prod_{j=1}^{n} S_j}{n!\mu^n}\right)^{\mu/A_n} \xrightarrow{d} e^{(\Gamma(\alpha+1))^{1/\alpha}\mathscr{L}}, \tag{2.2}$$

where $\Gamma(\alpha+1) = \int_0^\infty x^\alpha e^{-x}\,dx$.

Remark. When $\alpha = 2$, X is said to be in the domain of attraction of the normal. A special case is when $E(X^2) < \infty$. Since $S(x) \to \sigma^2 = \mathrm{Var}(X)$ as $x \to \infty$, $A_n \sim \sigma\sqrt{n}$. Moreover, $\Gamma(\alpha+1) = \Gamma(3) = 2$. Therefore, our result coincides with that of Rempała and Wesołowski (2002).

Before we proceed to the proof of the theorem we give the following lemma.

Lemma 2.3. *Under the conditions of Theorem* 2.2

$$\sum_{j=1}^{n} \frac{\log(\frac{n+1}{j})}{A_n}(X_j - \mu) \xrightarrow{d} (\Gamma(\alpha+1))^{1/\alpha}\mathscr{L}.$$

Proof. We unify the proofs for the cases $\alpha < 2$ and $\alpha = 2$. First let $A(x) = U(x)$ and $D(x) = 1/(1-F(x))$ if $\alpha < 2$, and $A(x) = V(x)$ and $D(x) = x^2/S(x)$ if $\alpha = 2$. According to Loève (1977), when $\alpha < 2$, $1 - F(x)$ is regularly varying with index $-\alpha$, and thus $1/(1-F)$ is regularly varying with index α; when $\alpha = 2$, $S(x)$ is slowly varying and $x^2/S(x)$ is regularly varying with index 2. Therefore, we have that $D(x)$ is regularly varying with index $\alpha \in (1,2]$, and that $A(x)$ is the generalized inverse of $D(x)$ and is regularly varying with index $1/\alpha \in [1/2, 1)$. (See, e.g., Bingham et al., 1987, Theorem 1.5.12). Moreover,

$$D(A(x)) \sim A(D(x)) \sim x \quad \text{as } x \to \infty. \tag{2.3}$$

Let f denote the characteristic function of $X_i - \mu$. Then (2.1) is equivalent to the following convergence

$$n\left(1 - f\left(\frac{t}{A(n)}\right)\right) \to -\log\phi(t) \quad \text{locally uniformly over } t \in R.$$

(See, e.g., Bingham et al., 1987, Lemma 8.2.0). Since $A(x)$ is regularly varying, we have as $x \to \infty$

$$x\left(1 - f\left(\frac{t}{A(x)}\right)\right) \to -\log\phi(t) \quad \text{locally uniformly over } t \in R.$$

Then substituting x by $D(s)$ we have from (2.3) the following consequence: as $s \to \infty$

$$D(s)\left(1 - f\left(\frac{t}{s}\right)\right) \to -\log\phi(t) \quad \text{locally uniformly over } t \in R. \tag{2.4}$$

Let f_n denote the characteristic function of $\sum_{j=1}^{n} \frac{\log(\frac{n+1}{j})}{A_n}(X_j - \mu)$. Then

$$f_n(t) = \prod_{j=1}^{n} f\left(\frac{\log(\frac{n+1}{j})}{A_n}t\right) = \prod_{j=1}^{n} f\left(\frac{t}{A_n/\log((n+1)/j)}\right).$$

We will show that for any $t \in R$,

$$\lim_{n\to\infty} f_n(t) = (\phi(t))^{\Gamma(\alpha+1)}. \tag{2.5}$$

First, note that $A_n/\log((n+1)/j) \geqslant A_n/\log(n+1)$ for all $1 \leqslant j \leqslant n$, and

$$\liminf_{n\to\infty} \frac{A_n}{\sqrt{n}} > 0. \tag{2.6}$$

Then, from (2.4) we get for any fixed t

$$D(A_n/\log((n+1)/j))\left(1 - f\left(\frac{t}{A_n/\log((n+1)/j)}\right)\right) \to -\log\phi(t)$$

uniformly for $1 \leqslant j \leqslant n$, or equivalently

$$f\left(\frac{t}{A_n/\log((n+1)/j)}\right) = 1 + \frac{1 + o(1)}{D(A_n/\log((n+1)/j))}\log\phi(t)$$

where $o(1)$ is a term that tends to 0 uniformly over j as n tends to infinity. Without any further notice, we use $o(1)$ in the sequel to denote some term with such a property. Since $1 + x = e^{x+o(x)}$ as $x \to 0$, we get

$$f\left(\frac{t}{A_n/\log((n+1)/j)}\right) = \exp\left\{\frac{1 + o(1)}{D(A_n/\log((n+1)/j))}\log\phi(t)\right\}$$

and hence we conclude for any fixed $t \in R$

$$f_n(t) = \exp\left\{(1 + o(1))\log\phi(t)\sum_{j=1}^{n}\frac{1}{D(A_n/\log((n+1)/j))}\right\}.$$

Therefore, in order to show (2.5), it suffices to prove

$$\sum_{j=1}^{n} \frac{1}{D(A_n/\log((n+1)/j))} \to \Gamma(\alpha+1). \tag{2.7}$$

Since $D(x)$ is regularly varying with index α, it can be written as $D(x)=x^{\alpha}L(x)$, where $L(x)$ is slowly varying. Then

$$\frac{1}{D(A_n/\log((n+1)/j))} = \frac{(\log((n+1)/j))^{\alpha}}{D(A_n)} \frac{L(A_n)}{L(A_n/\log((n+1)/j))}.$$

From (2.3), $D(A_n)=D(A(n)) \sim n$. Thus,

$$\sum_{j=1}^{n} \frac{1}{D(A_n/\log((n+1)/j))} = (1+o(1)) \sum_{j=1}^{n} \frac{(\log((n+1)/j))^{\alpha}}{n} \frac{L(A_n)}{L(A_n/\log((n+1)/j))}.$$

For any fixed small $\delta > 0$, we will break the right-hand side of the above equation into three sums:

$$\begin{aligned}
&\sum_{j=1}^{n} \frac{(\log((n+1)/j))^{\alpha}}{n} \frac{L(A_n)}{L(A_n/\log((n+1)/j))} \\
&= \sum_{\delta n \leqslant j \leqslant (1-\delta)n} \frac{(\log((n+1)/j))^{\alpha}}{n} \frac{L(A_n)}{L(A_n/\log((n+1)/j))} \\
&\quad + \sum_{1 \leqslant j < \delta n} \frac{(\log((n+1)/j))^{\alpha}}{n} \frac{L(A_n)}{L(A_n/\log((n+1)/j))} \\
&\quad + \sum_{(1-\delta)n < j \leqslant n} \frac{(\log((n+1)/j))^{\alpha}}{n} \frac{L(A_n)}{L(A_n/\log((n+1)/j))} \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

By properties of slow variation, as $s \to \infty$, $L(sx)/L(s) \to 1$ uniformly over $x \in C$, where C is any compact interval in $(0,\infty)$. Therefore,

$$\mathrm{I} = (1+o(1)) \sum_{\delta n \leqslant j \leqslant (1-\delta)n} \frac{(\log((n+1)/j))^{\alpha}}{n} \to \int_{\delta}^{1-\delta} (-\log x)^{\alpha}\,\mathrm{d}x.$$

To estimate II and III, we need the Potter's bound for slowly varying function $L(x)$: $L(x)/L(y) \leqslant \max(x/y, y/x)$ when x and y are sufficiently large (Bingham et al., 1987, Theorem 1.5.6). Hence, we get

$$\mathrm{II} \leqslant \sum_{1 \leqslant j < \delta n} \frac{(\log((n+1)/j))^{\alpha+1}}{n} \to \int_{0}^{\delta} (-\log x)^{\alpha+1}\,\mathrm{d}x,$$

and

$$\mathrm{III} \leqslant \sum_{\delta n < j \leqslant n} \frac{(\log((n+1)/j))^{\alpha-1}}{n} \to \int_{1-\delta}^{1} (-\log x)^{\alpha-1}\,\mathrm{d}x.$$

So we get for all small $\delta > 0$

$$\limsup_{n\to\infty}\left|\sum_{j=1}^{n}\frac{(\log((n+1)/j))^{\alpha}}{n}\frac{L(A_n)}{L(A_n/\log((n+1)/j))}-\int_0^1(-\log x)^{\alpha}\,\mathrm{d}x\right|$$

$$\leqslant 2\int_0^{\delta}(-\log x)^{\alpha+1}\,\mathrm{d}x+2\int_{1-\delta}^{1}(-\log x)^{\alpha-1}\,\mathrm{d}x,$$

which tend to 0 as $\delta\to 0$. Therefore,

$$\sum_{j=1}^{n}\frac{(\log((n+1)/j))^{\alpha}}{n}\frac{L(A_n)}{L(A_n/\log((n+1)/j))}\to\int_0^1(-\log x)^{\alpha}\,\mathrm{d}x=\int_0^{\infty}x^{\alpha}\mathrm{e}^{-x}\,\mathrm{d}x.$$

That proves (2.7) and (2.5).

Finally, we need to show that $\phi(t)^{\Gamma(\alpha+1)}$ is the characteristic function of $\Gamma(\alpha+1)^{1/\alpha}\mathscr{L}$. For $\alpha\in(1,2]$, it is obvious from the expression of ϕ that $\phi(t)^{\Gamma(\alpha+1)}=\phi(\Gamma(\alpha+1)^{1/\alpha}t)$. That completes the proof of the lemma. □

Proof of Theorem 2.2. It is easily seen that for some constant $c\in(0,\infty)$

$$\prod_{j=1}^{n}\frac{\log(1+\frac{1}{j})}{\frac{1}{j}(1-\frac{1}{2j})}\to c,$$

which, coupled with (2.6), yields

$$\left(\prod_{j=1}^{n}\frac{\log(1+\frac{1}{j})}{\frac{1}{j}}\right)^{\mu/A_n}\to 1.$$

Thus, it suffices to show

$$\left(\prod_{j=1}^{n}\frac{\log(1+\frac{1}{j})S_j}{\mu}\right)^{\mu/A_n}\xrightarrow{d}\mathrm{e}^{(\Gamma(\alpha+1))^{1/\alpha}\mathscr{L}}.$$

By strong law of large numbers, with probability one

$$\varepsilon_j=:\frac{\log(1+\frac{1}{j})S_j}{\mu}-1=j\log\left(1+\frac{1}{j}\right)\frac{S_j-j\mu}{j\mu}+j\log\left(1+\frac{1}{j}\right)-1\to 0 \tag{2.8}$$

as $j\to\infty$.

Notice that $E|X|^r<\infty$ for all $1<r<\alpha$. For our purpose, we fix $r\in(2\alpha/(\alpha+1),\alpha)$. We have by Marcinkiewicz-Zygmund's strong law of large numbers (see, eg., Chow and Teicher, 1988, p. 125), $S_j-j\mu=\mathrm{o}(j^{1/r})$ almost surely. Then $((S_j-j\mu)/j\mu)^2=\mathrm{o}(j^{2/r-2})$ almost surely. Therefore,

$$\sum_{j=1}^{n}\left[j\log\left(1+\frac{1}{j}\right)\right]^2\left(\frac{S_j-j\mu}{j\mu}\right)^2=\mathrm{o}(n^{2/r-1})$$

and

$$\sum_{j=1}^{n}\left[j\log\left(1+\frac{1}{j}\right)-1\right]^2=\mathrm{O}(\log n).$$

We conclude that with probability one

$$\sum_{j=1}^{n}\varepsilon_j^2 \leqslant 2\sum_{j=1}^{n}\left[j\log\left(1+\frac{1}{j}\right)\right]^2\left(\frac{S_j-j\mu}{j\mu}\right)^2+2\sum_{j=1}^{n}\left[j\log\left(1+\frac{1}{j}\right)-1\right]^2=\mathrm{o}(n^{2/r-1}).$$

Since $2/r-1<1/\alpha$ and $A(x)$ is regularly varying with index $1/\alpha$, and thus

$$\lim_{x\to\infty} x^{2/r-1}/A(x)=0$$

from property of regular variation. So we get

$$\frac{\sum_{j=1}^{n}\varepsilon_j^2}{A_n}\to 0 \tag{2.9}$$

with probability one.

In view of (2.8),

$$\frac{\log(1+\frac{1}{j})S_j}{\mu}=1+\varepsilon_j=\mathrm{e}^{\varepsilon_j+\mathrm{O}(\varepsilon_j^2)},$$

and thus from (2.9)

$$\left(\prod_{j=1}^{n}\frac{\log(1+\frac{1}{j})S_j}{\mu}\right)^{\mu/A_n}=\exp\left\{\frac{\mu\sum_{j=1}^{n}\varepsilon_j}{A_n}+\mathrm{O}\left(\frac{\sum_{j=1}^{n}\varepsilon_n^2}{A_n}\right)\right\}=\exp\left\{\frac{\mu\sum_{j=1}^{n}\varepsilon_j}{A_n}+\mathrm{o}(1)\right\}.$$

The remaining task is to show that

$$\frac{\mu\sum_{j=1}^{n}\varepsilon_j}{A_n}\xrightarrow{d}(\Gamma(\alpha+1))^{1/\alpha}\mathscr{L}. \tag{2.10}$$

As a matter of fact,

$$\begin{aligned}\frac{\mu\sum_{j=1}^{n}\varepsilon_j}{A_n}&=\frac{1}{A_n}\sum_{j=1}^{n}\log\left(1+\frac{1}{j}\right)(S_j-j\mu)+\frac{\mu}{A_n}\sum_{j=1}^{n}\left(j\log\left(1+\frac{1}{j}\right)-1\right)\\&=\sum_{j=1}^{n}\frac{\log(\frac{n+1}{j})}{A_n}(X_j-\mu)+\mathrm{O}\left(\frac{\log n}{A_n}\right)\\&=\sum_{j=1}^{n}\frac{\log(\frac{n+1}{j})}{A_n}(X_j-\mu)+\mathrm{o}(1).\end{aligned}$$

(2.10) is proved by applying Lemma 2.3. That completes the proof. □

3. Open problem

One would be interested in asymptotic behavior of the product of sums when X is in the domain of attraction of a stable law when the index $\alpha\in(0,1]$. Unfortunately we are unable to prove whether (2.2) holds for $\alpha\in(0,1]$. We guess (2.2) could be true when $\alpha=1$ but $E(X)<\infty$. For case $\alpha\in(0,1)$ (this implies $E(X)=\infty$) or $\alpha=1$ but $E(X)=\infty$, one has to find some appropriate normalization constants for the product of the sums. This is an unsolved problem as well.

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