Hotelling’s Model

- Hot Dog Vendors on Rockefeller Plaza

Monopoly ($N = 1$)

- Production level $\bar{y}$ was given.
- What if $\bar{y}$ is endogenous and firm’s location choice affects profit of other firms?
Notes

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- Hotelling’s model (c.f. any advanced (or some intermediate) microeconomics textbook like Varian Chpt 25 [Var05]).
- Think of a region in which consumers are uniformly located along a line segment $[0, 1]$.
- Each consumer prefers to travel a shorter distance to a hot dog vendor (homogeneous good).
- There are $N$ sellers.
- Where would we expect these sellers to choose their locations?

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Notes

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- Consumer location is given by $x$ ($0 \leq x \leq 1$).
- $n$-th vendor’s location is given by $x^n$ ($0 \leq x^n \leq 1$).
- Mill price of $n$-th vendor is given by $p^n$.
- Delivered price of consumer at $x$ is given by

$$p^n + \frac{|x - x^n|}{\text{distance to vendor } n}.$$
Each consumer buys one hot dog.

- Normalize $p^n = 1$ for all $n$.
- Demand faced by Liz is $D^L(x^L, x^K, \ldots)$ hot dogs.
- Demand faced by Kenneth is $D^K(x^L, x^K, \ldots)$ hot dogs.
- (Assume the cost is sunk, i.e., the vendor has already made enough hot dogs).
- Liz’s profit is $1 \cdot D^L(x^L, x^K, \ldots) = D^L(x^L, x^K, \ldots)$ dollars.
- Kenneth’s profit is $1 \cdot D^K(x^L, x^K, \ldots) = D^K(x^L, x^K, \ldots)$ dollars.

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### Question 2.1 (Location of the Monopolist)

1. What is the demand that the monopolist faces?
2. What is the monopolist’s profit when she locates $x = 0.25$ and $0.5$?
3. Which location maximizes the vendor’s profit?
4. Which location minimizes total delivered price paid by the customers?
Liz solves:

$$\max_{0 \leq x^L \leq 1} \pi^j (x^L) = 1 \cdot D^j (x^L).$$

$$D^j (x^L) = 1.$$  

$$\pi^j (x^L) = 1 \cdot D^j (x^L) = 1.$$  

Liz’s profit is 1 regardless of the location $x^L$. 

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**Notes**

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Delivered price is location-variant.

Total delivered price is vendor’s profit and the aggregate sum of $|x - x^L|$ from $x = 0$ to $x = 1$.

\[
TDP(x^L = 0) = 1 + .5 = 1.5 = \frac{24}{16}
\]

\[
TDP(x^L = .25) = 1 + \frac{5}{16} = \frac{21}{16}
\]

\[
TDP(x^L = .5) = 1 + \frac{1}{2} = \frac{5}{4} = \frac{20}{16}
\]
What is socially optimal for the economy as a whole is $x^* = .5$ (minimizes total cost of trip while maintaining the profitability).

What is privately optimal for the vendor is any $x^*$ in $[0, 1]$.

**Question 2.2 (Profit Maximization and Social Welfare)**

- Is there any way that Liz chooses $x^* = .5$ by herself?
- Is political intervention necessary to realize $x^* = .5$?

Suppose there are two vendors ($N = 2$).

Liz solves:

$$\max_{0 \leq x^L \leq 1} \pi^L(x^L, x^K) = 1 \cdot D^L(x^L, x^K).$$

Kenneth solves:

$$\max_{0 \leq x^K \leq 1} \pi^K(x^L, x^K) = 1 \cdot D^K(x^L, x^K).$$

Note Liz does not solve:

$$\max_{0 \leq x^L \leq 1, 0 \leq x^K \leq 1} \pi(x^L, x^K) = 1 \cdot D(x^L, x^K).$$

(why?)

What do $D^L(x^L, x^K)$, $D^K(x^L, x^K)$ look like?

Start with $(x^L, x^K) = (0, 1)$.
Model

- Monopoly
- Duopoly
- Nash Equilibrium
- Oligopoly

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Notes

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\[ D^L(0, 1) = 0.5 \Rightarrow \pi^L(0, 1) = 0.5. \]

\[ D^K(0, 1) = 0.5 \Rightarrow \pi^K(0, 1) = 0.5. \]

Does Liz improve her profit by moving to the east?
\[ D_L(1, 1) = \frac{1 + 1}{2} = .55 \]
\[ D_K(1, 1) = 1 - .55 = .45. \]

Does Kenneth improve his profit by moving to the west?

- \[ D_L(1, .8) = \frac{1 + .8}{2} = .45. \]
- \[ D_K(1, .8) = 1 - .45 = .55. \]
If $x^L < x^K$, Liz takes the consumers from 0 to the midpoint between $x^L$ and $x^K$:

$$D_L(x^L, x^K) = \frac{x^L + x^K}{2}.$$  

Kenneth takes the rest:

$$D_K(x^L, x^K) = 1 - D_L(x^L, x^K) = 1 - \frac{x^L + x^K}{2}.$$  

If $x^L > x^K$, Kenneth takes the consumers from 0 to the midpoint between $x^L$ and $x^K$:

$$D_K(x^L, x^K) = \frac{x^L + x^K}{2}.$$  

Liz takes the rest:

$$D_L(x^L, x^K) = 1 - D_K(x^L, x^K) = 1 - \frac{x^L + x^K}{2}.$$
An isoprofit curve of Liz indicates the set of location 
($x^L$, $x^K$) that results in the same profit.

For Liz, isoprofit at .4 is a set of location ($x^L$, $x^K$) 
that results in $\pi^L(x^L, x^K) = .4$.

- If $x^L < x^K$ (region above 45° line) 
  
  \[ \pi^L(x^L, x^K) = \frac{x^L + x^K}{2} = .4 \Rightarrow x^K = -x^L + .8. \]

- If $x^L > x^K$ (region below 45° line) 
  
  \[ \pi^L(x^L, x^K) = 1 - \frac{x^L + x^K}{2} = .4 \Rightarrow x^K = -x^L + 1.2. \]

- If $x^L = x^K$ (45° line) 
  
  \[ \pi^L(x^L, x^K) = .5 \neq .4 \]
For Liz, isoprofit at .5 is a set of location $(x^L, x^K)$ that results in $\pi^L(x^L, x^K) = .5$.

1. If $x^L < x^K$ (region above 45° line)
   \[ \pi^L(x^L, x^K) = \frac{x^L + x^K}{2} = .5 \Rightarrow x^K = -x^L + 1. \]

2. If $x^L > x^K$ (region below 45° line)
   \[ \pi^L(x^L, x^K) = 1 - \frac{x^L + x^K}{2} = .5 \Rightarrow x^K = -x^L + .2. \]

3. If $x^L = x^K$ (45° line)
   \[ \pi^L(x^L, x^K) = .5 \]

A crossbuck shape.

For Kenneth, isoprofit at .4 is a set of location $(x^L, x^K)$ that results in $\pi^K(x^L, x^K) = .4$.

1. If $x^L < x^K$ (region below 45° line)
   \[ \pi^K(x^L, x^K) = 1 - \frac{x^L + x^K}{2} = .4 \Rightarrow x^K = -x^L + 1.2. \]

2. If $x^L > x^K$ (region above 45° line)
   \[ \pi^K(x^L, x^K) = \frac{x^L + x^K}{2} = .4 \Rightarrow x^K = -x^L + .8. \]

3. If $x^L = x^K$ (45° line)
   \[ \pi^K(x^L, x^K) = .5 \neq .4 \]

Question: where is Kenneth’s isoprofit at .5 located?
How does Liz react to Kenneth's location?

If Kenneth is at $x^K = 0$, Liz maximizes its profit by locating slightly to the east of $x^K$.

$(x^L, 0) \Rightarrow x^L = 0 + \epsilon$ (where $\epsilon$ is an infinitesimally small distance).

If Kenneth is at $x^K = .2$, Liz maximizes its profit by locating slightly to the east of $x^K$.

$(x^L, .2) \Rightarrow x^L = .2 + \epsilon$.

If Kenneth is at $x^K = .4$, Liz maximizes its profit by locating slightly to the east of $x^K$.

$(x^L, .4) \Rightarrow x^L = .4 + \epsilon$.

$(x^L, .5) \Rightarrow x^L = .5$.

$(x^L, .6) \Rightarrow x^L = .6 - \epsilon$.

$(x^L, .8) \Rightarrow x^L = .8 - \epsilon$.

$(x^L, 1) \Rightarrow x^L = 1 - \epsilon$. 
How does Kenneth react to Liz’s location?

1. \((0, x^L) \Rightarrow x^K = 0 + \varepsilon\).
2. \((.2, x^L) \Rightarrow x^K = .2 + \varepsilon\).
3. \((.4, x^L) \Rightarrow x^K = .4 + \varepsilon\).
4. \((.5, x^L) \Rightarrow x^K = .5\).
5. \((.6, x^L) \Rightarrow x^K = .6 - \varepsilon\).
6. \((.8, x^L) \Rightarrow x^K = .8 - \varepsilon\).
7. \((1, x^L) \Rightarrow x^K = 1 - \varepsilon\).

Suppose that vendors take turns in changing locations (in the denomination of .01 miles).

1. Given \((x^L, x^K) = (0, 1)\), Liz locates slightly to the west of Kenneth: \((.99, 1)\).
2. Given \((x^L, x^K) = (.99, 1)\), Kenneth locates slightly to the west of Liz: \((.99, .98)\).

...
Nash equilibrium is the location \((x^{\text{NE}}, x^{\text{NE}})\) such that none of the vendor can profit by unilaterally changing its location.

For duopoly, the Nash equilibrium \((x^{\text{NE}}, x^{\text{NE}}) = (0.5, 0.5)\).

Is the Nash equilibrium \((x^{\text{NE}}, x^{\text{NE}}) = (0.5, 0.5)\) socially efficient?

Fact: the socially efficient allocation is \((x^*, x^*) = (1/4, 3/4)\) or \((3/4, 1/4)\).
**Nash Equilibrium & Socially Efficient Location**

### Compare:

<table>
<thead>
<tr>
<th>Location</th>
<th>Profit</th>
<th>TDP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nash (.5, .5)</td>
<td></td>
<td>1+1/4</td>
</tr>
<tr>
<td>Socially Efficient</td>
<td>(.25, .75)</td>
<td>1+1/8</td>
</tr>
<tr>
<td>Socially Efficient</td>
<td>(.75, .25)</td>
<td>1+1/8</td>
</tr>
</tbody>
</table>

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**Liz’s Isoprofit**

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**Notes**

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Question 4.1 (Socially Inefficient Equilibrium)

Why can’t vendors choose the socially optimal location, when both Nash equilibrium location and socially efficient location yield the same profits?

They are indifferent between \((x^{NE}, x^{KNE})\) and \((x^*, x^{K*})\).

Given \((x^L, x^K) = (0.25, 0.75)\), Liz will move to a location slightly to the west of \(x^K = 0.75\).

\((x^L, x^K) = (0.74, 0.75) \rightarrow (0.74, 0.73) \rightarrow (0.72, 0.73) \rightarrow \cdots \rightarrow (0.5, 0.5)\)
If there are 3 vendors, Liz solves
\[
\max_{0 \leq x \leq 1} \pi(x, x^L, x^D) = 1 \cdot D^L(x^L, x^K, x^D).
\]
For \( N = 3 \), there is no Nash equilibrium.
Consider
1. All the vendors locate at different place.
2. \( x^L = x^K = x^D \).
3. \( x^L = x^K \neq x^D \).

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Fact: socially efficient location is $(x^L, x^K, x^D) = (1/6, 3/6, 5/6)$ (not necessarily in this order).
There are Nash equilibria for $N \geq 4$. 

- Hotelling’s Model
  - Hot Dog Vendors on Rockefeller Plaza
- Monopoly ($N = 1$)
- Duopoly ($N = 2$)
  - Duopoly
  - Isoprofit
- Nash Equilibrium
  - Nash Equilibrium & Socially Efficient Location
- Oligopoly ($N \geq 2$)
  - $N = 3$
  - $N \geq 4$

Summary

Notes

- Social optimality vs individual optimality.
- Isoprofit
- Nash equilibrium
References


Map du Jour

http://www.worldmapper.org/