Chapter 1

Special Relativity

Problem Set #1: 1.1, 1.3, 1.7, 1.10, 1.13 (Due Monday Sept. 23rd)

1.1 Minkowski space

In Newtonian physics the three spatial dimensions x, y and z are connected by coordinate transformations, yet the time coordinate t was always treated separately. For example one can rotate the Cartesian coordinate system without altering the distance

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 . \tag{1.1}$$

There are six linearly independent transformations of space: three shifts and three rotations. However, until special theory of relativity there were no useful definitions of invariant distance in space-time described by all four coordinates t, x, y and z. We knew that the time coordinate must be different from space, but the connection was not clear until the following proposal for the invariant distance was made

$$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = -(c\Delta t')^2 + (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 . \tag{1.2}$$

It is more convenient to set $c = 1$ and to use the upper indices notation

$$t \rightarrow x^0 \tag{1.3}$$
$$x \rightarrow x^1 \tag{1.4}$$
$$y \rightarrow x^2 \tag{1.5}$$
$$z \rightarrow x^3 . \tag{1.6}$$

(Note that upper indices should not to be confused with exponentiating!) Then with the **Einstein summation convention** (i.e. always sum over repeated upper and lower indices) we obtain

$$(\Delta s)^2 = \sum_{\mu,\nu=0,1,2,3} \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu \equiv \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu . \tag{1.7}$$

What are the transformation which leave the distance in space-time invariant? For example if we change

$$x^\mu \to x^{\mu'} + a^\mu \tag{1.8}$$

the invariant distance defined by (1.8) would not change. These are the shifts in space a^1, a^2, a^3 and in time a^0. What about other transformations analogous to rotations? Consider

$$x^\mu \to x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu \tag{1.9}$$

where Λ is some 4×4 matrix such that in matrix notation $x' = \Lambda x$. For the distance (1.7) to be invariant we must have

$$(\Delta s)^2 = (\Delta x)^T \eta\, (\Delta x) = (\Delta x')^T \eta\, (\Delta x') = (\Delta x)^T \Lambda^T \eta \Lambda\, (\Delta x) \tag{1.10}$$

and thus,

$$\eta = \Lambda^T \eta \Lambda \tag{1.11}$$

or

$$\eta_{\rho\sigma} = \Lambda^{\mu'}{}_\rho \Lambda^{\nu'}{}_\sigma\, \eta_{\mu'\nu'} . \tag{1.12}$$

Such transformations are known as **Lorentz transformations** and the group such transformations is called **Lorentz group** or $O(3,1)$.

There are six generators of O(3,1): the three usual rotations and three Lorentz boosts. For example,

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.13}$$

describes rotation where the angle $\theta \in [0, \pi]$

$$\Lambda = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{1.14}$$

describes boosts where the parameter $\phi \in (-\infty, +\infty)$. Nevertheless, it is still useful to think of boosts as rotations between space and time. Altogether how many linearly independent continuous transformations leave the distance (1.7) invariant? one time shift + three space shifts + three rotations + three boosts = ten! the group of such transformation is a non-abelian group known as the **Poincare group**. There are also discrete reflections of each of four coordinates, but they are not a part of the Poincare group.

The new ingredient in special relativity (that you must have seen before) are the boosts which correspond to changing coordinates to a moving frame. From (1.14) the transformed coordinates are

$$t' = t\cosh\phi - x\sinh\phi \tag{1.15}$$
$$x' = -t\sinh\phi + x\cosh\phi. \tag{1.16}$$

Thus, in the original coordinate system the point corresponding to $x' = 0$ or

$$-t\sinh\phi + x\cosh\phi = 0 \tag{1.17}$$

is moving with velocity

$$v \equiv \frac{x}{t} = \tanh\phi. \tag{1.18}$$

Then the transformation laws (1.15) and (1.16) can be written in a more familiar form

$$t' = t\cosh\left(\tanh^{-1} v\right) - x\sinh\left(\tanh^{-1} v\right) = \gamma\,(t - vx) \tag{1.19}$$
$$x' = -t\sinh\left(\tanh^{-1} v\right) + x\cosh\left(\tanh^{-1} v\right) = \gamma(x - vt) \tag{1.20}$$

where $\gamma = 1/\sqrt{1-v^2}$, but you should check this.

It is also easy to see what the boost transformation do to your axes $x = 0$ and $y = 0$ using the space-time diagram. The transformed axes x'=0 and $t' = 0$ in terms of the old coordinates are described by lines

$$x = t\,\tanh\phi \tag{1.21}$$
$$x = \frac{t}{\tanh\phi}. \tag{1.22}$$

The only paths that are the same in both coordinates systems are the ones described by light rays, i.e. $x = \pm t$ is the same as $x' = \pm t'$. The set of all light rays emitted from a given space-time point is called the **light cone**. This set divides all of the points into light-like (or **null**) separated, space-like separated, future time-like separated and past time-like separated.

1.2 Vectors and one-forms

Vector space is a set of objects which can be added together or multiplied by a real number in a linear way, i.e.

$$(a + b)(V + W) = aV + bV + aW + bW. \tag{1.23}$$

The elements in a vector space are called vectors. The **basis** is a set of linearly independent vectors (i.e. none of the basis vectors can be expressed as a linear combination of remaining basis vectors) that span all of the vector space (i.e. any vector in the vector space can be expressed as a linear combination of basis vectors). The dimensionality of the vector space is the maximum number of linearly independent vectors or the number of basis vectors that we shall denote by $\hat{e}(\mu)$.

In general relativity the number of dimensions is four, corresponding to one time $\mu = 0$ and three spatial dimensions $\mu = 1, 2, 3$, but in more exotic theories (with extra dimensions) the number can be arbitrary large. The vectors in a four diminutional space-time of general relativity are called **four-vectors**. As in the case of electrodynamics the four-vectors are located at a given point and do not stretch from one point to another. The space of all possible vectors at a given point p is called the **tangent space** denoted by T_p. For example, the tangent space of a given point on a two-dimensional sphere is a two-dimensional plane. The set of all tangent spaces of a given manifold is called a **tangent bundle**, $T(M)$. For example, a four-vector field is an element of a tangent bundle.

Any four-vector can be written as a linear combination of four (not necessarily orthogonal) basis vectors,

$$A = A^\mu \hat{e}_{(\mu)}. \tag{1.24}$$

It is a standard practice in the literature to refer to components A^μ as a four-vector when the actual four-vector is A. Of course, $\hat{e}(\mu)$ is a collection of four basis vectors and not components of some vector.

Consider a curve $x^\mu(\lambda)$ through space-time parametrized by λ. At every point on the curve one can define a **tangent vector**

$$V = V^\mu \hat{e}_{(\mu)} = \frac{dx^\mu(\lambda)}{d\lambda} \hat{e}_{(\mu)} \tag{1.25}$$

with components

$$V^\mu = \frac{dx^\mu(\lambda)}{d\lambda}. \tag{1.26}$$

In the case when trajectories are not straight lines it is useful to introduce a concept of infinitesimal proper distance or a line element

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu.$$

which measures a proper distance along a curve through space-time $x^\mu(\lambda)$ parametrized by λ

$$\Delta s = \begin{cases} \int \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda & \text{for spacelike curves} \\ \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda & \text{for timelike curves.} \end{cases}$$

(If the curve $x^\mu(\tau)$ describes a trajectory of some particle of mass m and τ is the proper time along the trajectory, then V^μ is usually also called a **four-velocity** and mV^μ a **four-momentum** of the particle whose zeroth component is the energy of the particle. In this case V^μ is normalized, $V_\nu V^\nu = -1$.)

Under Lorentz transformation the components change as

$$V^\mu \to V^{\mu'} = \Lambda^{\mu'}{}_\nu V^\nu, \tag{1.27}$$

but the four-vector V is, of course, invariant, i.e.

$$V = V^\mu \hat{e}_{(\mu)} = V^{\nu'} \hat{e}_{(\nu')} = \Lambda^{\nu'}{}_\mu V^\mu \hat{e}_{(\nu')} \tag{1.28}$$

Since the above expression is true for all V^μ , the basis vectors must also transfer as

$$\hat{e}_{(\mu)} = \Lambda^{\nu'}{}_\mu \hat{e}_{(\nu')}. \tag{1.29}$$

By denoting an inverse matrix as

$$\Lambda_\mu{}^\nu \equiv \left(\Lambda^{-1}\right)^\nu{}_\mu \tag{1.30}$$

such that

$$\Lambda_{\nu'}{}^\mu \Lambda^{\sigma'}{}_\mu = \delta^{\sigma'}_{\nu'},\ \Lambda^{\nu'}{}_\rho \Lambda_{\nu'}{}^\mu = \delta^\mu_\rho, \tag{1.31}$$

we find that the basis vectors transform using the **inverse Lorentz transformation**

$$\hat{e}_{(\nu')} = \Lambda_{\nu'}{}^\mu \hat{e}_{(\mu)}. \tag{1.32}$$

We note that the invariant objects with all of the **dummy indices** contracted (such as $V = V^\mu \hat{e}_{(\mu)}$ here) will always remain invariant under appropriate transformations of the components and basis vectors.

For every tangent space T_p there is an associated vector space of the same dimensionality known as **cotangent space** T^*_p and for every tangent bundle

$T(M)$ there is an associated **cotangent bundle** $T^*(M)$. The cotangent space (or dual space or a space of one-forms) is defined as a space of linear maps from the tangent space to real numbers. Then every element of the dual space $\omega \in T_p^*$ is a one-from defined by a map

$$\omega : T_p \to \mathbb{R} \tag{1.33}$$

such that

$$\omega(aV + bW) = a\omega(V) + b\omega(W) \tag{1.34}$$

for every $a, b \in \mathbb{R}$ and $V, W \in T_p$. One can see that the dual space can be thought of as a vector space itself if

$$(a\omega + b\eta)(V) = a\omega(V) + b\eta(V) \tag{1.35}$$

with basis one-forms defined by

$$\hat{\theta}^{(\nu)}(\hat{e}_{(\mu)}) = \delta^\nu_\mu. \tag{1.36}$$

Then every one-form ω is a linear combination of dual basis vectors

$$\omega = \omega_\mu \hat{\theta}^{(\mu)}, \tag{1.37}$$

but in the literature (similarly to vectors) one often refers to the components of a one-form ω_μ as it is a one-form. In fact the basis vectors and basis one-forms are often omitted for the reason that the action of one-form on a vector is simply a real number,

$$\omega(V) = \omega_\mu \hat{\theta}^{(\mu)}(V^\nu \hat{e}_{(\nu)}) = \omega_\mu V^\nu \hat{\theta}^{(\mu)}(\hat{e}_{(\nu)}) = \omega_\mu V^\nu \delta^\mu_\nu = \omega_\mu V^\mu \in \mathbb{R}. \tag{1.38}$$

The transformation laws for the components of one-forms ω_μ and basis one-forms $\hat{\theta}^{(\mu)}$are trivially derived using the same arguments as for the vectors,

$$\omega_{\mu'} = \Lambda_{\mu'}{}^{\nu} \omega_\nu \tag{1.39}$$
$$\hat{\theta}^{(\rho)} = \Lambda^{\rho'}{}_{\sigma} \hat{\theta}^{(\sigma)}. \tag{1.40}$$

As a result the one-forms (1.37) as well as the scalar quantities such as (1.38) are invariant under Lorentz transformations.

Note that (1.38) suggests that one can think of vector as maps from the dual space to real numbers. It is also useful to think of vectors as a four component column vector and of one-forms as four component row vectors whose action on each other is a matrix multiplication. In fact one might have already noticed that the vectors and one-forms in general relativity are treated similarly to the ket and bra vectors in quantum mechanics with the difference that components in quantum mechanics are complex valued.

The simplest example of a one-form is the **gradient** of a scalar function ϕ denoted here by

$$\mathrm{d}\phi \equiv \frac{\partial \phi}{\partial x^{\mu}} \hat{\theta}^{(\mu)} \tag{1.41}$$

with components transformed according to chain rule

$$\frac{\partial \phi}{\partial x^{\mu'}} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial \phi}{\partial x^{\mu}} = \Lambda_{\mu'}{}^{\mu} \frac{\partial \phi}{\partial x^{\mu}}. \tag{1.42}$$

Thus, the gradient transforms as a one-form which motivates the use of the following shorthand notations

$$\frac{\partial \phi}{\partial x^{\mu}} = \partial_{\mu} \phi = \phi_{,\mu}. \tag{1.43}$$

One can act with the gradient (which is a one-form) on a partial derivative of a parametrized curve (which is a vector) to get an ordinary derivative of ϕ along the curve (which is an invariant quantity), i.e.

$$\partial_{\mu} \phi \frac{\partial x^{\mu}}{\partial \lambda} = \frac{d\phi}{d\lambda}. \tag{1.44}$$

1.3 Mathematics of tensors

As we have seen above, vectors and one-forms can be considered respectively as linear maps from the space of one-forms and vectors to real numbers, e.g.

$$V : T_p^* \rightarrow \mathbb{R} \tag{1.45}$$

and

$$\omega : T_p \rightarrow \mathbb{R}. \tag{1.46}$$

Of course one can also think about maps from a collection of k vectors and l one-forms to real numbers, e.g.

$$T : T_p^* \times ... \times T_p^* \times T_p \times ... \times T_p \rightarrow \mathbb{R}. \tag{1.47}$$

Such objects are called **tensors** of **rank** (k, l) and all of the tensors of a given rank form a vector space (i.e. can be added together or multiplied by a real number in a linear manner). More generally the tensors of rank $(k+m, l+n)$ can be used to define maps from tensors of rank (k, l) to tensors of rank (m, n) for arbitrary k, l, m and n.

In tensor terminology vectors are just $(0, 1)$ tensors, one-forms are $(1, 0)$ tensors and scalars (which are not really map, but real numbers) are called $(0, 0)$ tensors. Another tensor that we have already seen is a metric $\eta_{\mu\nu}$. It is a (0,2) **symmetric** tensor,

$$\eta_{\mu\nu} = \eta_{\nu\mu}. \tag{1.48}$$

whose action on a pair of vectors is called the **dot product**,

$$\eta(V, W) = \eta_{\mu\nu} V^\mu W^\nu = V \cdot W. \tag{1.49}$$

Then the two vectors are orthogonal if their inner product is zero, and since the inner product is a scalar it would remain zero in an arbitrary inertial frame. The norm of a vector is defined as an inner product of a vector with itself which must not be always positive in Minkowski space in contrast to Euclidean space. A somewhat surprising fact in Minkowski space is the existence of the so-called **null vectors** whose norm is zero, but not all of the components are zero.

Just like it is convenient to thing of vectors as column vectors and of one-forms as row vectors one may think of $(1, 1)$ tensors as square matrices whose action on vectors and one-forms is determined by matrix multiplication. For example, **Kronecker delta** function

$$\delta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.50}$$

is a symmetric $(1, 1)$ tensor. Similarly, more general tensors could the thought of as higher dimensional hyper-cubic matrices. For example, the $(0, 4)$ **Levi-Civita** tensor is a 4 dimensional hyper-cubic matrix with components given by

$$\epsilon_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \mu\nu\rho\sigma \text{ is an even premutations of } 0123 \\ -1 & \text{if } \mu\nu\rho\sigma \text{ is an even premutations of } 0123 \\ 0 & \text{otherwise.} \end{cases} \tag{1.51}$$

which is completely **antisymmetric** (i.e. under interchange of any two indices) tensor since

$$\epsilon_{\mu\nu\rho\sigma} = -\epsilon_{\nu\mu\rho\sigma} = -\epsilon_{\rho\nu\mu\sigma} = -\epsilon_{\sigma\nu\rho\mu}. \tag{1.52}$$

A completely antisymmetric tensor of rank $(0, p)$ is called a **differential p-form** which is why the $(0, 1)$ are called one-forms. Although not all of the tensors are symmetric (or antisymmetric) an arbitrary tensor can always symmetrize (or antisymmetrize) in any of its indices,

$$T_{(\mu_1...\mu_n)} \equiv \frac{1}{n!} \sum_{\text{all permutations of } \mu_1...\mu_n} T_{\mu...\mu_n} \tag{1.53}$$

and

$$T_{[\mu_1...\mu_n]} \equiv \frac{1}{n!} \left(\sum_{\text{even permutations of } \mu_1...\mu_n} T_{\mu...\mu_n} - \sum_{\text{odd permutations of } \mu_1...\mu_n} T_{\mu...\mu_n} \right). \tag{1.54}$$

Of course, the symmetrization (or antisymmetrization) does not have to be carried over all indices, e.g.

$$T_{[\mu|\nu\sigma|\rho]} = \frac{1}{2} \left(T_{\mu\nu\sigma\rho} - T_{\rho\nu\sigma\mu} \right). \tag{1.55}$$

One can also form a tensor of a higher rank $(k+m, l+n)$ from tensors of lower ranks (k,l) and (m,n) is by the so-called **tensor product**,

$$\begin{aligned} T \otimes S(\omega^{(1)}, ..., \omega^{(k+m)}, V^{(1)}, ..., V^{(l+n)}) = \\ = T(\omega^{(1)}, ..., \omega^{(k)}, V^{(1)}, ..., V^{(l)}) \quad S(\omega^{(k+1)}, ..., \omega^{(k+m)}, V^{(l+1)}, ..., V^{(l+n)}), \end{aligned} \tag{1.56}$$

where the superscripts in parentheses denote distinct vector and one-forms, but not their components. Clearly the tensor product is not commutative in general, i.e

$$T \otimes S \neq S \otimes T. \tag{1.57}$$

The basis vectors for the vector space of tensors of rank (k,l) can be written as a tensor product of k basis vectors and l basis one-forms,

$$\hat{e}_{(\mu_1)} \otimes ... \otimes \hat{e}_{(\mu_k)} \otimes \hat{\theta}^{(\nu_1)} \otimes ... \otimes \hat{\theta}^{(\nu_l)} \tag{1.58}$$

such that

$$T = T^{\mu_1...\mu_k}{}_{\nu_1...\nu_l} \hat{e}_{(\mu_1)} \otimes ... \otimes \hat{e}_{(\mu_k)} \otimes \hat{\theta}^{(\nu_1)} \otimes ... \otimes \hat{\theta}^{(\nu_l)} \tag{1.59}$$

but, as usual, one often refers to the components $T^{\mu_1...\mu_k}{}_{\nu_1...\nu_l}$ as to a full tensor of rank (k,l). This also implies that the transformation laws for component of tensor are exactly as one would naively expect,

$$T^{\mu'_1...\mu'_k}{}_{\nu'_1...\nu'_l} = \Lambda^{\mu'_1}{}_{\mu_1} ... \Lambda^{\mu'_k}{}_{\mu_k} \Lambda_{\nu'_1}{}^{\nu_1} ... \Lambda_{\nu'_l}{}^{\nu_l} T^{\mu_1...\mu_k}{}_{\nu_1...\nu_l}. \tag{1.60}$$

Moreover given two tensors of ranks $(0, p)$ and $(0, q)$ (or given two differential forms) on can define a **wedge product**

$$(A \wedge B)_{\mu_1 \dots \mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1 \dots \mu_p} B_{\mu_{p+1} \dots \mu_{p+q}]} \tag{1.61}$$

which is a tensor of rank $(0, p+q)$ or a $(p+q)$-form.

Some other standard operations with tensor include:

- **contraction of indices**, e.g.
$$S^{\mu\rho}{}_{\sigma} = T^{\mu\nu\rho}{}_{\sigma\nu} \tag{1.62}$$
- **raising of indices** using the inverse metric, e.g.
$$T^{\alpha\beta\gamma}{}_{\delta} = \eta^{\mu\gamma} T^{\alpha\beta}{}_{\mu\delta} \tag{1.63}$$
- **lowering of indices** using the metric, e.g
$$T^{\alpha}{}_{\beta\gamma\delta} = \eta_{\mu\beta} T^{\alpha\mu}{}_{\gamma\delta}, \tag{1.64}$$
- **Hodge dual** (or **star)** operation using the metric and Levi-Civita tensor, e.g.
$$(\star A)_{\mu_1 \dots \mu_{4-p}} = \frac{1}{p!} \epsilon^{\nu_1 \dots \nu_p}{}_{\mu_1 \dots \mu_{4-p}} A_{\nu_1 \dots \nu_p}. \tag{1.65}$$

As an aside note that although the gradient was introduced as a one-form one, $\partial_\mu \phi$, but can define a more familiar gradient vector by raising its indices, i.e. $\eta^{\mu\nu} \partial_\nu \phi$. One might now wonder if we could act with a partial derivative operator on an arbitrary tensor of rank (k, l) to form a tensor of rank $(k, l+1)$, e.g.

$$T_{\alpha}{}^{\mu}{}_{\nu} = \partial_\alpha R^{\mu}{}_{\nu}? \tag{1.66}$$

The answer is positive in Minkowski space of special relativity, but not on more general geometries of general relativity (although the gradient of a scalar is always a legitimate $(0, 1)$ tensor). On the other hand the so-called **exterior derivative** of an arbitrary p-form

$$(\mathrm{d}A)_{\mu_1 \dots \mu_{p+1}} \equiv (p+1) \partial_{[\mu_1} A_{\mu_2 \dots \mu_{p+1}]} \tag{1.67}$$

is a $(0, p+1)$ tensor in any space-time and form any $p \geq 0$. From commutativity of partial derivatives one can also show that

$$\mathrm{d}\,(\mathrm{d}A) = 0. \tag{1.68}$$

Thus, all $p-$forms A are **closed** (i.e. $\mathrm{d}A = 0$) if they are **exact** (i.e. $A = \mathrm{d}B$), but the opposite is not always true (although it is true for $p > 0$ in any space with topology $\mathbb{R}^4$ such as Minkowski space).

1.4 Physics of tensors

Let us consider an antisymmetric $(0,2)$ **electromagnetic field tensor**,

$$F_{\mu\nu} = -F_{\nu\mu} \tag{1.69}$$

whose component are made out of the electric and magnetic fields

$$F = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{1.70}$$

with a Hodge dual

$$\star F = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{pmatrix} \tag{1.71}$$

Both F and $\star F$ transforms under the Lorentz transformations exactly as a $(0,2)$ tensor should. If we also define a four-vector current from an electric current $\mathbf{J}$ and charge J^0 densities, i.e.

$$J = \begin{pmatrix} \rho \\ J_1 \\ J_2 \\ J_3 \end{pmatrix}, \tag{1.72}$$

then the Maxwell equations

$$\nabla \times \mathbf{B} - \partial_t \mathbf{E} = \mathbf{J} \tag{1.73}$$
$$\nabla \cdot \mathbf{E} = \rho \tag{1.74}$$
$$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0 \tag{1.75}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{1.76}$$

written in terms of components

$$\epsilon^{ijk}\partial_j B_k - \partial_0 E^i = J^i \tag{1.77}$$
$$\partial_i E^i = J^0 \tag{1.78}$$
$$\epsilon^{ijk}\partial_j E_k + \partial_0 B^i = 0 \tag{1.79}$$
$$\partial_i B^i = 0 \tag{1.80}$$

or in terms of field tensor

$$\begin{aligned}
\partial_j F^{ij} + \partial_0 F^{i0} &= J^i && (1.81)\\
\partial_j F^{0j} &= J^0 && (1.82)\\
\partial_j \left(\star F^{ij}\right) + \partial_0 \left(\star F^{i0}\right) &= 0 && (1.83)\\
\partial_j \left(\star F^{0j}\right) &= 0. && (1.84)
\end{aligned}$$

This can be rewritten as only two equations

$$\begin{aligned}
\partial_\mu F^{\nu\mu} &= J^\nu && (1.85)\\
\partial_\mu \star F^{\nu\mu} &= 0 && (1.86)
\end{aligned}$$

or in terms of exterior derivative

$$\begin{aligned}
\mathrm{d} \star F &= \star J && (1.87)\\
\mathrm{d} F &= 0. && (1.88)
\end{aligned}$$

It is remarkable to note that in Minkowski space equation (1.88) implies the existence if a one-form A such that

$$F = \mathrm{d}A \tag{1.89}$$

which is the familiar four-vector potential,

$$A = (\phi, A_1, A_2, A_3). \tag{1.90}$$

One can also use the Lorentz symmetry to derive an expression for the **Lorentz force** on a particle of charge q and four-velocity v^μ is a four-vector given by

$$f^\mu = qV^\nu F^\mu{}_\nu \tag{1.91}$$

which reduces to the familiar expression from small velocities

$$\mathbf{f} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \tag{1.92}$$

as one would expect. For a stationary particle of mass m the four-momentum was defined as

$$p^\mu = (m, 0, 0, 0) \tag{1.93}$$

and in a boosted frame in x-direction the four-momentum is given by

$$p^\mu = (\gamma m, v\gamma m, 0, 0) \tag{1.94}$$

such that $p_\mu p^\mu = -m^2$ is invariant and

$$f^\mu = m\frac{d^2}{d\tau^2}x^\mu(\tau) = \frac{d}{d\tau}p^\mu. \tag{1.95}$$

Although the four-velocity is a valid description for a single particle, in the case of many particles it is difficult to keep track of all of the degrees of freedom. In such a limit one usually describes the system in terms of fluids or of coarse-grained fields of particles characterized by density, pressure, etc. It turns out that the four-vector momentum field is not sufficient to describe many properties of particles and one must define tensor field such as energy-momentum $(2, 0)$ tensor, $T^{\mu\nu}$. Note, that for a fluid of strings, in addition to a symmetric tensor field $T^{\mu\nu}$, one also needs to introduce an antisymmetric tensor field $F^{\mu\nu}$ just to make the fluid equations consistent. One (but not the only) way to define the energy-momentum tensor is as a flux of the-four momentum field p^μ across a surface of constant x^ν.

The simplest of all fluids is a **perfect fluid** which (in addition to the four-velocity V^μ) is characterized by two parameters: energy density ρ and pressure p,

$$T^{\mu\nu} = (\rho + p)V^\mu V^\nu + p\eta^{\mu\nu}. \tag{1.96}$$

often related to each other by the so-called **equations of state parameter**

$$w = \frac{p}{\rho}. \tag{1.97}$$

For simple matter (such as stars or dark matter) $w = 0$, and simple radiation (such as photons or gravity waves) $w = \frac{1}{3}$. A little bit more exotic fluids of non-interacting cosmic strings and domain walls have $w = -\frac{1}{3}$ and $-\frac{2}{3}$ respectively, but with interaction the number would be somewhat smaller. So far the discussion of fluids was somewhat phenomenological, but very often one want to derive it the energy momentum tensor from a theory starting from the variational principle. Very soon we will be able show that the energy momentum tensor is nothing but a variation of the Lagrangian density with respect to metric.

$$T^{\mu\nu} = \frac{\delta\mathcal{L}}{\delta g_{\mu\nu}} \tag{1.98}$$

As we will see the energy momentum tensor for a massive scalar field (e.g. Higgs field) is

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\left(\frac{1}{2}\partial_\lambda\phi\partial^\lambda\phi + V(\phi)\right) \tag{1.99}$$

and for a massless vector field (e.g. electromagnetic field) is

$$T^{\mu\nu} = F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}\eta^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}. \tag{1.100}$$

Using the equations of motion one can show the energy momentum tensor defines conserved quantities, i.e.

$$\partial_\mu T^{\mu\nu} = 0. \tag{1.101}$$

For a perfect fluid (1.96) this implies

$$\partial_\mu T^{\mu\nu} = \partial_\mu(\rho + p)V^\mu V^\nu + (\rho + p)\left(\partial_\mu V^\mu V^\nu + V^\mu \partial_\mu V^\nu\right) + \partial^\nu p \tag{1.102}$$

But since $V_\mu V^\mu = -1$,

$$\partial_\mu\left(V_\mu V^\mu\right) = 2V_\nu \partial_\mu V^\nu = 0 \tag{1.103}$$

and

$$V_\nu \partial_\mu T^{\mu\nu} = -\partial_\mu\left(\rho V^\mu\right) - p\partial_\mu V^\mu. \tag{1.104}$$

In the Newtonian (i.e. $V^\mu = (1, V^1, V^2, V^3)$ where$V^i \ll 1$) and (small pressure $p \ll \rho$) limit we get a continuity equation

$$\partial_0 \rho + \nabla \cdot (\rho \mathbf{V}) = 0. \tag{1.105}$$

One can also project (1.101) to a direction orthogonal to V^μ to, i.e. using

$$P^{\sigma}{}_{\nu} = \delta^\sigma_\nu + V^\sigma V_\nu \tag{1.106}$$

to obtain in the Newtonian limit the Euler equation of fluid mechanics

$$\rho(\partial_0 \mathbf{V} + (\mathbf{V} \cdot \nabla)\mathbf{V}) = -\nabla p. \tag{1.107}$$

1.5 Classical Field Theory

In classical mechanics an action of a single particle described by coordinates $q(t)$ is

$$S[q] \equiv \int dt\, L(q, \dot{q}). \tag{1.108}$$

which can be thought of as a functional of $q(t)$. Then the classical equations of motion can be obtained from the most important principle in physics - the variational principle:

$$\frac{\delta S[q]}{\delta q} = 0. \tag{1.109}$$

By Taylor expanding the Lagrangian

$$L(q + \delta q) \approx L(q) + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \tag{1.110}$$

we obtain

$$\begin{aligned} 0 = \frac{\delta S}{\delta q} &= \int dt \left(\frac{\delta L}{\delta q}\right) \\ &= \frac{\int dt \, (L(q+\delta q) - L(q))}{\delta q} \\ &= \frac{\int dt \, \left(\frac{\partial L(q)}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}\right)}{\delta q} \\ &= \frac{\int dt \, \left(\frac{\partial L(q)}{\partial q}\delta q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q\right)}{\delta q}, \end{aligned} \tag{1.111}$$

or up to a boundary term

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \tag{1.112}$$

For example,

$$L = \frac{1}{2}\dot{q}^2 - V(q) \tag{1.113}$$

gives rise to

$$\ddot{q} = -\frac{\partial V}{\partial q}. \tag{1.114}$$

In classical field theory an action for a collection of fields described by $\Phi^i(x^0, x^1, x^2, x^3)$ is

$$S[\Phi^1, ..., \Phi^N] = \int d^4x \, \mathcal{L}(\Phi^1, \partial_0\Phi^1, \partial_1\Phi^1, \partial_2\Phi^1, \partial_3\Phi^1, ...) \tag{1.115}$$

(which is a slightly more complicated functional) one can still use the variational principle to obtain N equations of motion

$$\frac{\delta S}{\delta \Phi^i} = 0 \tag{1.116}$$

for N degrees of freedom. Note that the action is dimensionless which suggests that the so-called Lagrangian density $\mathcal{L}$ must have the dimensions

$$[\mathcal{L}] = [Length]^{-4} = [Time]^{-4} = [Mass]^4 = [Energy]^4. \tag{1.117}$$

By Taylor expanding the perturbed Lagrangian

$$\mathcal{L}(..., \Phi^i+\delta\Phi^i, \partial_\mu\Phi^i+\partial_\mu\Phi^i, ...) = \mathcal{L}(..., \Phi^i, \partial_\mu\Phi^i, ...)+\frac{\partial\mathcal{L}}{\partial\Phi^i}\delta\Phi^i+\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Phi^i\right)}\delta\left(\partial_\mu\Phi^i\right) \tag{1.118}$$

we get (up to the boundary term)

$$0 = \frac{\delta S}{\delta\Phi^i} = \frac{\int d^4x \left[\frac{\partial\mathcal{L}}{\partial\Phi^i} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^i)}\right)\right]\delta\Phi^i}{\delta\Phi^i} \tag{1.119}$$

or

$$\frac{\partial\mathcal{L}}{\partial\Phi^i} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Phi^i\right)}\right) = 0. \tag{1.120}$$

For example, the scalar field Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{1.121}$$

gives rise to an equation of motion

$$\Box\phi - \frac{\partial V}{\partial\phi} = 0, \tag{1.122}$$

where

$$\Box \equiv \partial_\mu\partial^\mu. \tag{1.123}$$

For a massless vector field coupled to a conserved current, i.e.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu \tag{1.124}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.125}$$

the equations of motion are

$$\partial_\mu F^{\nu\mu} = J^\nu. \tag{1.126}$$