2.3 Lecture 7: Binary collisions

1. Without collisions:

   (a) A neighborhood \( dqdp \) moves to \( dq'dp' = dqdp \) since for Hamiltonian systems the flow is incompressible.

   (b) A molecule at \((q, p, t)\) moves to \((q + \frac{p}{m}\delta t, p + F\delta t, t + \delta t)\) in time \(\delta t\), i.e.

   \[
   f(q + \frac{p}{m}\delta t, p + F\delta t, t + \delta t) dqdp' = f(q, p, t) dqdp
   \]
   or

   \[
   f(q + \frac{p}{m}\delta t, p + F\delta t, t + \delta t) = f(q, p, t)
   \]

2. With collisions:

   \[
   f(q + \frac{p}{m}\delta t, p + F\delta t, t + \delta t) = f(q, p, t) + \left( \frac{\partial f}{\partial t} \right)_{\text{coll}} \delta t
   \]

By expanding the left hand site in powers of \(\delta t\) and keeping only linear terms we get the transport equation

\[
\frac{\partial}{\partial t} f(q, p, t) + \frac{p}{m} \cdot \frac{\partial}{\partial q} f(q, p, t) + F \cdot \frac{\partial}{\partial p} f(q, p, t) = \left( \frac{\partial f}{\partial t} \right)_{\text{coll}}
\]

which is not very useful unless \(\left( \frac{\partial f}{\partial t} \right)_{\text{coll}}\) is specified. Of course, finding or modeling the collision term is the biggest challenge in the kinetic theory. In the simplest model one only takes into account binary collisions and assumes that the colliding particles are uncorrelated (i.e. molecular chaos assumption).

Consider an elastic collision of two spherically symmetric (spin-less) molecules with \(m_1, p_1\) and \(m_2, p_2\). After collision their respective momenta are \(p'_1\) and \(p'_2\). Then the following conservation laws apply:

- Momentum conservation:

  \[
  p_1 + p_2 = p'_1 + p'_2.
  \]

- Energy conservation:

  \[
  \frac{p'^2_1}{m_1} + \frac{p'^2_2}{m_2} = \frac{p^2_1}{m_1} + \frac{p^2_2}{m_2}.
  \]
Define:

- Reduced mass
  \[ \mu = \frac{m_1 m_2}{m_1 + m_2} \]  
  (2.38)

- Relative momentum
  \[ \mathbf{p} = \frac{m_1 \mathbf{p}_1 + m_2 \mathbf{p}_2}{m_1 + m_2} = \mu \left( \frac{\mathbf{p}_1}{m_1} + \frac{\mathbf{p}_2}{m_2} \right) \]  
  (2.39)

Then the conservation of energy can be written as

\[ |\mathbf{p}| = |\mathbf{p}'|, \]  
(2.40)
i.e. collision simply rotates relative momentum. Such rotations can be specified by a unit vector in spherical coordinates \((\theta, \phi)\), where

- inclination angle \(\theta\) between \(\mathbf{p}\) and \(\mathbf{p}'\)
- azimuthal angle \(\phi\) of \(\mathbf{p}'\) about \(\mathbf{p}\).

The dynamical aspect of the collisions are described by differential cross section \(|d\sigma/d\Omega|\) - the cross-sectional area which scatters particles into solid angle \(d\Omega\) around \(\Omega\). For scattering of two spheres of diameter \(D\) with impact parameter \(b\) the scattering angle is given by

\[ \cos \left( \frac{\theta}{2} \right) = \frac{b}{D} \]  
(2.41)

and thus, the cross-sectional area is

\[ d\sigma = bd\phi db = D \cos \left( \frac{\theta}{2} \right) D \sin \left( \frac{\theta}{2} \right) d\theta d\phi = \frac{D^2}{4} \sin \theta d\theta d\phi = \frac{D^2}{4} d\Omega \]  
(2.42)

and

\[ \frac{d\sigma}{d\Omega} = \frac{D}{4} \]  
(2.43)

The total cross-section is given by

\[ \sigma_{\text{total}} = \int \left| \frac{d\sigma}{d\Omega} \right| d\Omega \]  
(2.44)

which

\[ \sigma_{\text{total}} = \int \frac{D^2}{4} d\Omega = \pi D^2 \]  
(2.45)
for an elastic scattering of hard spheres.

In quantum mechanics the scattering probabilities \( \mathcal{P}_{p_1, p_2 \rightarrow p_1', p_2'} \) per unit time per unit volume are expressed through scattering amplitudes \( A_{p_1, p_2 \rightarrow p_1', p_2'} \) using Born rule,

\[
\mathcal{P}_{p_1, p_2 \rightarrow p_1', p_2'} = \delta(p_1+p_2-p_1'-p_2') \delta(p_1'^2+p_2'^2-p_1^2-p_2^2) \left| A_{p_1, p_2 \rightarrow p_1', p_2'} \right|^2
\]

\[
= \delta(p_1+p_2-p_1'-p_2') \delta(p_1'^2+p_2'^2-p_1^2-p_2^2) \left| \langle p_1', p_2' | T | p_1, p_2 \rangle \right|^2 (2.46)
\]

where \( T \) is the quantum mechanical transition matrix and all of the factors of \((2\pi)\) where absorbed into definition of \( A_{p_1, p_2 \rightarrow p_1', p_2'} \). If we assume that the interactions are invariant under spatial rotations, reflections and time reversal (e.g. electromagnetic interactions), then the transition matrix \( T \) has the following symmetries

\[
\langle p_1', p_2' | T | p_1, p_2 \rangle = \langle R p_1', R p_2' | T | R p_1, R p_2 \rangle
\]

\[
\langle p_1', p_2' | T | p_1, p_2 \rangle = \langle -p_1, -p_2 | T | -p_1', -p_2' \rangle, \quad (2.47)
\]

where \( R \) is a matrix representing spatial rotations and/or reflections. Note that the vector \( p \) can also include spin d.o.f., then \( R \) rotates the spatial and spin d.o.f., but reflects only spatial d.o.f. It follows form (2.47) that

\[
\langle p_1', p_2' | T | p_1, p_2 \rangle = \langle p_1, p_2 | T | p_1', p_2' \rangle. \quad (2.48)
\]

This can be shown by:

- time reversal

\[
\langle p_1', p_2' | T | p_1, p_2 \rangle = \langle -p_1, -p_2 | T | -p_1', -p_2' \rangle,
\]

- rotation on angle \( \pi \) around any axis perpendicular to total momentum,

\[
\langle p_1', p_2' | T | p_1, p_2 \rangle = \langle -R_2 p_1', -R_2 p_2' | T | -R_2 p_1, -R_2 p_2 \rangle,
\]

- reflection to complete the interchange the initial and final relative momenta,

\[
\langle p_1', p_2' | T | p_1, p_2 \rangle = \langle -R_2 R_2 p_1', -R_2 R_2 p_2' | T | -R_2 R_2 p_1, -R_2 R_2 p_2 \rangle = \langle p_1, p_2 | T | p_1', p_2' \rangle,
\]

One can apply the ideas of kinetic theory to other systems with a large number of d.o.f. For example, a very large network of cosmic strings can be described by a transport equation analogous to the Boltzmann transport equation for particles.

**Question to Go:** Think of a system with a large number of d.o.f. for which the ideas of the kinetic theory might be useful.