Math 3280 Practice Midterm 2 Solutions

(1) Find the general solution to the ODE: $y^{(3)} - 5y'' + 12y' - 8y = 0$.

Solution: The characteristic equation is $r^3 - 5r^2 + 12r - 8$. If we believe in a benevolent testwriter, it is natural to look for integer solutions to polynomials of degree larger than two. So we could try $1, -1, 2, -2, 4, -4, 8, -8$. Happily it is easy to check that 1 is a root, so the characteristic polynomial has $(r-1)$ as a factor. After dividing out this factor (you should know how to do polynomial division!) we get $r^2 - 4r + 8$. From the quadratic equation we can then find the full factorization $(r-1)(r-(2-2i))(r-(2+2i))$. The general solution is $y = C_1 e^x + e^{2x}(C_2 \sin(2x) + C_3 \cos(2x))$

(2) Find the solution to the initial value problem $y'' - 2y' + 5y = e^{2x}$, $y'(0) = 0$, $y(0) = -1$.

Solution: We begin by finding the general solution $y = y_h + y_p$. The homogeneous solution y_h is determined by the characteristic equation $r^2 - 2r + 5 = (r-(1+2i))(r-(1-2i))$: $y_h = e^x(C_1 \cos(2x) + C_2 \sin(2x))$.

We can find the particular solution y_p by the method of undetermined coefficients, i.e. we suppose that $y_p = Ae^{2x}$ and solve for A. Plugging in this form and dividing out the e^{2x} factors we find that $4A - 4A + 5A = 1$, or $A = 1/5$.

Now the initial conditions can be used to determine C_1 and C_2. The condition $y(0) = -1$ becomes $C_1 + \frac{1}{5} = -1$ and $y'(0) = 0$ becomes

$$2e^x C_2 \cos(2x) + e^x C_1 \cos(2x) + e^x C_2 \sin(2x) - 2e^x C_1 \sin(2x) + \frac{2e^{2x}}{5}|_{x=0}$$

$$= \frac{2}{5} + C_1 + 2C_2 = 0.$$

The first equation can be immediately solved for $C_1 = -\frac{6}{5}$ and then the second for $C_2 = \frac{2}{5}$. So the solution is $y = e^x(-\frac{6}{5}\cos(2x) + \frac{2}{5}\sin(2x)) + \frac{1}{5}e^{2x}$.

(3) Write down the form of a particular solution y_p of the ODE $y'' + y = x^2 e^x + \cos(x)$. You do not have to determine the coefficients of the functions.

Solution: The problem is a little harder than it might look because one of the functions on the righthand side also appears in the homogeneous solution $y_h = C_1 \cos x + C_2 \sin x$. So we have to add a power of x in the undetermined particular solution: $y_p = Ax\cos x + Bx \sin x + Ce^x + Dxe^x + Ex^2 e^x$.

(4) If an $n \times n$ matrix A has the property that $A^3 = 2A$, what are the possible values of the determinant of A?

Solution: Taking the determinant of both sides of the equation gives us $det(A^3) = det(2A)$. Because of the multiplicative property of determinants, $det(A^3) = (det(A))^3$.

Since each row of $2A$ has been multiplied by 2, $det(2A) = 2^n det(A)$. Then we have

$$(det(A))^3 - 2^n det(A) = det(A)((det(A))^2 - 2^n) = 0$$

so either $det(A) = 0$ or $det(A) = \pm 2^{n/2}$.

(5) Solve the initial value problem $y''' - 27y = e^{3x}$, $y(0) = y'(0) = y''(0) = 0$.
Solution:
First we find the homogeneous (also called complementary) solution to

$$y_c''' - 27y_c = 0.$$

To do this we have to factor the characteristic equation $r^3 - 27 = 0$.
One root is easy to get: $r_1 = (27)^{1/3} = 3$.
If we divide $r^3 - 27$ by $r - 3$, the quotient is $r^2 + 3r + 9$.
With the quadratic formula we can get the other two roots, $r_2, r_3 = -\frac{3}{2} \pm \frac{3\sqrt{3}i}{2}$. With these three roots, we can construct the complementary solution:

$$y_c = C_1 e^{3x} + C_2 e^{-\frac{3t}{2}} \cos(\frac{3\sqrt{3}t}{2}) + C_3 e^{-\frac{3t}{2}} \sin(\frac{3\sqrt{3}t}{2})$$

Next, to find the particular solution we would normally use the method of undetermined coefficients with the form $y_p = Ae^{3x}$.
But this is contained within the complementary solution, so instead we use

$$y_p = Axe^{3x}.$$

Since $y_p''' = 27xAe^{3x} + 27Ae^{3x}$, we require that

$$\begin{aligned} y_p''' - 27y_p &= 27xAe^{3x} + 27Ae^{3x} - 27xAe^{3x} \\ &= 27Ae^{3x} = e^{3x} \end{aligned}$$

and so $A = 1/27$.
So the general solution to the ODE is

$$y = y_c + y_p = C_1 e^{3x} + C_2 e^{-\frac{3t}{2}} \cos\left(\frac{3\sqrt{3}t}{2}\right) + C_3 e^{-\frac{3t}{2}} \sin\left(\frac{3\sqrt{3}t}{2}\right) + \frac{1}{27} x e^{3x}$$

The initial condition $y(0) = 0$ becomes $C_1 + C_2 = 0$. Since

$$y' = -\frac{3}{2}(\sqrt{3}C_2 + C_3) e^{-\frac{3x}{2}} \sin\left(\frac{3\sqrt{3}}{2}x\right) + \frac{3}{2}(\sqrt{3}C_3 - C_2) e^{-\frac{3x}{2}} \cos\left(\frac{3\sqrt{3}}{2}x\right) + (3\,C_1 + \frac{1}{27} + \frac{x}{9}) e^{3x}$$

$$y'(0) = \frac{3}{2}\sqrt{3}C_3 - \frac{3}{2}C_2 + 3\,C_1 + \frac{1}{27} == 0$$

Now we compute the equation for the initial condition $y''(0) = 0$

$$y'' = \frac{9}{2}e^{-\frac{3x}{2}}\left((\sqrt{3}C_2 - C_3)\sin\left(\frac{3\sqrt{3}}{2}x\right) + (\sqrt{3}C_3 + C_2)\cos\left(\frac{3\sqrt{3}}{2}x\right)\right) + e^{3x}\left(9C_1 + \frac{2}{9} + \frac{x}{3}\right)$$

$$y''(0) = -\frac{9}{2}\sqrt{3}C_3 - \frac{9}{2}C_2 + 9\,C_1 + \frac{2}{9} = 0$$

Writing all of these initial conditions as a matrix-vector system we get:

$$\begin{pmatrix} 1 & 1 & 0 \\ 3 & -\frac{3}{2} & \frac{3}{2}\sqrt{3} \\ 9 & -\frac{9}{2} & -\frac{9}{2}\sqrt{3} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/27 \\ -2/9 \end{pmatrix}$$

The row-reduced echelon form of the augmented coefficient matrix is

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{1}{81} \\ 0 & 1 & 0 & \frac{1}{81} \\ 0 & 0 & 1 & \frac{1}{243}\sqrt{3} \end{pmatrix}$$

So finally we have:

$$y = \frac{1}{81}\left[e^{-\frac{3x}{2}}\left(\frac{\sqrt{3}}{3}\sin\left(\frac{3\sqrt{3}}{2}x\right) + \cos\left(\frac{3\sqrt{3}}{2}x\right)\right) + (3x-1)\,e^{3x}\right]$$

(6) Find a basis for the subspace S of solutions to the system within the vector space $\{(x_1, x_2, x_3, x_4, x_5, x_6) |\ x_i \in \mathbb{R}\} = \mathbb{R}^6$:

$$x_1 - x_2 + x_4 + 4x_5 = 0$$
$$x_1 + x_2 + x_4 + 4x_5 + x_6 = 0$$

Solution:
We row reduce the coefficient matrix:

$$\begin{pmatrix} 1 & -1 & 0 & 1 & 4 & 0 \\ 1 & 1 & 0 & 1 & 4 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 4 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 1/2 \end{pmatrix}$$

This shows that x_1 and x_2 are pivot variables, while the others are free. We write the solution in terms of the free variables:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -x_4 - 4x_5 - x_6/2 \\ -x_6/2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} =$$

$$x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -4 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} -1/2 \\ -1/2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

So a basis for S is the set

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1/2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(7) Rewrite the initial value problem $y''' + y'' + y = t$, $y(0) = y'(0) = y''(0) = 0$ as an equivalent first-order system.

Solution: Introduce the variables $v_1 = y'$, $v_2 = v_1' = y''$ and the system becomes:

$$\begin{gathered} y' = v_1 \\ v_1' = v_2 \\ v_2' = t - v_2 - y \\ y(0) = 0,\ v_1(0) = 0,\ v_2(0) = 0 \end{gathered}$$

Note that rewriting the initial conditions is a required part of this answer.

(8) The matrix

$$A = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

where a and b are real numbers, is diagonalizable, i.e. there exists a matrix P such that $P^{-1}AP = D$ where D is diagonal. Compute D.

Solution:

$$D = \begin{pmatrix} a + bi & 0 & 0 \\ 0 & a - bi & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(any other order of the eigenvalues on the diagonal is also correct).

(9) Indicate whether each of the following statements is true or false.

(a) The set of solutions $(x, y, z) \in \mathbb{R}^3$ to the equation $x + y + z = 0$ is a vector subspace of $\mathbb{R}^3$ of dimension 2.

Solution: True. A single linear homogeneous constraint will have a solution set that is one dimension less than the ambient vector space. Alternatively we can compute this by row-reducing the coefficient matrix of the system, which in this case is the matrix $[1, 1, 1]$. This is already in row-reduced echelon form, with one pivot and two free variables (y and z). The number of free variables is the dimension of the solution set.

(b) The set of solutions $(x, y, z) \in \mathbb{R}^3$ to the equation $x + y = 1$ is a vector subspace of $\mathbb{R}^3$ of dimension 2.
Solution: False. This is a nonhomogeneous system, so the solutions do not form a vector subspace.

(c) The set of solutions to the differential equation $y'' + xy' + x^2 y = 0$ is a vector space of dimension 2.

Solution: True. See Theorem 4 of section 5.2.

(d) The set of solutions $(x, y, z) \in \mathbb{R}^3$ of the system below is a vector subspace of $\mathbb{R}^3$ of dimension 1.

$$\begin{aligned} x + 2y + 3z &= 0 \\ 4x + 5y + 6z &= 0 \\ 7x + 8y + 9z &= 0 \end{aligned}$$

Solution: True. The coefficient matrix has a row-reduced form with two pivots and one free variable.

(e) The polynomials $1+x$, $1-x$, $1+x^2$ are a basis for the vector space of polynomials with real coefficients of degree less than or equal to 2.

Solution: True. A more obvious basis would be $1, x, x^2$, which can be obtained from these polynomials as linear combinations: $1 = (1 + x)/2 + (1 - x)/2$, $x = (1 + x)/2 - (1 - x)/2$, and $x^2 = -(1 + x)/2 - (1 - x)/2 + (1 + x^2)$.