Concave Central Configurations in the Four-Body Problem

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Abstract

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Central configurations are important landmarks in the n-body problem. The history of their study is summarized, and it is proved that for any four positive masses there exists a concave central configuration with those masses. This question was first posed in a 1932 paper by MacMillan and Bartky [MB]. One corollary of the proof is that the set of equivalence classes of concave central configurations under rotations, dilations, reflections, and relabelings is homeomorphic to a three-dimensional ball (including only part of its boundary). For the generic case in which all the masses are different, another corollary is that there are at least eight concave central configurations with those masses.

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Chapter 1

INTRODUCTION

The study of the dynamics of n point masses interacting according to Newtonian gravity [N] is usually referred to as the n-body problem. More precisely, we consider n particles at $q_i \in \mathbb{R}^d$ with masses $m_i \in \mathbb{R}_+$, $i = 1, \ldots, n$, and $\ddot{q}_i = \sum_{i \neq j} m_j \frac{q_j - q_i}{|q_j - q_i|^3}$. There are many problems associated with the dynamics of such a system. The study of these problems has been immensely fruitful; it has motivated and influenced such developments as differential and integral calculus, convergence of series expansions, and chaotic dynamics. Many natural questions arise which prove difficult or impossible to solve, especially as the value of n is increased. The two-body problem has been completely solved, but already for $n = 3$ the complexity increases to such an extent that many open problems remain. Newton once wrote that he only got headaches when working on the three-body problem [We].

In order to make progress against such complexity one must ask relatively simple questions, make assumptions about the parameters of the system, or both. The most successful instance of the first strategy has been the study of the periodic orbits of the system. Besides being of interest in their own right, they are, as Poincaré wrote, "the only breach by which we can penetrate a fortress hitherto considered inaccessible" [Po1]. A particularly interesting type of periodic orbit in the planar n-body problem is one in which the particles remain in the same shape relative to one another. The shapes possible for the particles in such orbits are called *central configurations*; this term is apparently due to Wintner [Wn]. The equations determining central configurations can be generalized to define them in higher dimensions as well; more precise definitions will be given in Chapter 2.

After surveying some of the problems in which central configurations arise we will summarize some of the most important results concerning them. Euler and Lagrange classified all the central configurations of the three-body problem [Eu], [La], namely for any given masses there are three collinear configurations and the equilateral triangle. The four-body central configurations are poorly understood except in the case of all the masses being equal, which was recently (1996) solved by Albouy [Al2]. A few general tools will be summarized which have been developed to study the general case, but very little is known for $n \geq 4$. Then the theorems and lemmas needed to prove the main theorem of the paper will be presented. The main theorem is:

Theorem 1. *For any four positive masses* $(m_1, m_2, m_3, m_4) \in \mathbb{R}^4_+$ *there exists a planar concave central configuration with the fourth point strictly in the convex hull of the other three points.*

The proof consists of several steps. After defining a canonical representative for each equivalence class of concave configurations under rotations, dilations, reflections, and re-labelings, the set of such representatives of concave central configurations is shown to be homeomorphic to a three-dimensional ball (including only part of its boundary). Next, a map from the interior of this set to the space of masses $\mathcal{M}$ is extended to the boundary. In order to define a continuous extension, some of the boundary points must be blown up. Finally, the extended map restricted to the boundary is shown to have non-zero degree onto its image, which implies that the map is surjective onto $\mathcal{M}$. A more detailed outline of the proof and some of its corollaries is given in chapter 13.

Chapter 2

IMPORTANCE OF CENTRAL CONFIGURATIONS

After making the definition of central configurations more precise, this chapter will establish some notation and conventions, and briefly sketch the important applications of central configurations.

In the n-body problem we consider n particles at $q_i \in \mathbb{R}^d$ with masses $m_i \in \mathbb{R}_+$, $i = 1, \ldots, n$, and the dynamics given by $\ddot{q}_i = \sum_{j\neq i} m_j \frac{q_j - q_i}{|q_j - q_i|^3} = \frac{1}{m_i}\frac{\partial U}{\partial q_i}$, where U is the Newtonian potential $U = \sum_{i<j} \frac{m_i m_j}{|q_i - q_j|}$. Note that this potential function is positive, in contrast to the convention often used in introductory books. We will use $q \in \mathbb{R}^{nd}$ and $m \in \mathbb{R}^n_+$ to denote the position and mass vectors $(q_1, \ldots, q_n)$ and $(m_1, \ldots, m_n)$ respectively. Because the total momentum of the particles is conserved, one can split the n-body dynamics into a linear motion of the center of mass $c = \frac{1}{M}\sum m_i q_i$ and motion around the center of mass, where $M = \sum_{i=1}^n m_i$. So without loss of generality we will assume that the center of mass is at the origin (for details on this procedure see [SM]). To avoid singularities we will restrict q to be in $\mathbb{R}^{nd} \setminus \Delta$, where $\Delta = \bigcup_{i\neq j}\{q|\ q_i = q_j\}$.

Definition 1. A position vector $q \in \mathbb{R}^{nd} \setminus \Delta$ is defined to be a central configuration if there exists a mass vector $m \in \mathbb{R}^n_+$ and a parameter $\lambda \in \mathbb{R}_+$ such that

$$\lambda q_i = \sum_{j\neq i} m_j \frac{q_i - q_j}{|q_i - q_j|^3}. \tag{2.1}$$

Since the right hand side of (2.1) is equal to $-\ddot{q}_i$, in the dynamical context each particle is accelerated toward the center of mass in proportion to its distance from the center of mass. It is not hard to show that if every particle in such a configuration starts with an initial velocity of zero the particles will collapse to the origin, reaching it simultaneously.

Configurations (vectors in $\mathbb{R}^{nd} \setminus \Delta$) are considered equivalent if they are similar modulo reflections and relabelings, i.e. $q, p \in \mathbb{R}^{nd} \setminus \Delta$ are equivalent if there is a rotation and dilation of $\mathbb{R}^d$ that sends each q_i to $p_{\sigma(i)}$, where σ is a permutation of $\{1, 2, 3, 4\}$. Another way to phrase this is that the equivalence classes are the orbits of $\mathbb{R}^+ \cdot O(d) \times S_4$ acting on $(\mathbb{R}^d)^n$ by $(\lambda R, \sigma)(q_1, q_2, q_3, q_4) = (\lambda R q_{\sigma^{-1}(1)}, \lambda R q_{\sigma^{-1}(2)}, \lambda R q_{\sigma^{-1}(3)}, \lambda R q_{\sigma^{-1}(4)})$. It is easy to see from the homogeneity and symmetry of (2.1) that any configuration which is equivalent to a central configuration is also a central configuration. By the number of central configurations we will mean the number of equivalence classes under the above equivalence. Sometimes it will be more convenient to speak of affine equivalence classes, which are the same as above but without relabelings or reflections - i.e. without the S_4 action and with $S0(d)$ instead of $O(d)$.

There is another characterization of central configurations that is often useful, which is due to Dziobek [D]. For a given mass vector, if we consider the set of positions S with a moment of inertia I equal to 1, $S = \{q \in \mathbb{R}^{nd} \setminus \Delta \mid \sum m_i |q_i|^2 = 1 \text{ and } \sum m_i q_i = 0\}$, then the fixed points under the gradient flow $X = \nabla(U|_S)$ are central configurations. The gradient is defined via the metric induced by the Euclidean metric on $\mathbb{R}^d$. The restriction to $I = 1$ removes the dilational degeneracy; in order to obtain isolated fixed points the rotational degeneracy must be removed by considering the quotient of S under the diagonal action of $SO(d)$ as described above. The gradient flow descends to the quotient space and then there is a one to one correspondence between fixed points of the flow and affine equivalence classes of central configurations with the given mass vector.

In the planar case, which is the focus of this paper, it is sometimes notationally convenient to use complex coordinates. Following the presentation in [SM], we write the position vector as $z \in \mathbb{C}^n \setminus \Delta$, where $(Re[z_i], Im[z_i]) = q_i$. Now consider the homographic solutions of the planar n-body problem, i.e. solutions which may rotate and dilate about the origin but which remain similar to the initial configuration. Such solutions can be written as $z(t) = \zeta(t)w$, where $\zeta(t)$ is a complex function of the real variable t, and $w \in \mathbb{C}^n$. Plugging this form of a solution into Newton's equations gives us $\ddot{\zeta} w_i = \zeta|\zeta|^{-3} \sum_{j \neq i} \frac{m_j(w_j - w_i)}{|w_j - w_i|^3}$ for $i = 1, \ldots, n$. If we combine terms that do not depend on t we obtain (2.1) in complex coordinates, where $-\lambda < 0$ is the constant that the time dependent terms must equal, i.e. we also have $\ddot{\zeta}\zeta^{\frac{1}{2}}\bar{\zeta}^{\frac{3}{2}} = -\lambda$.

So in order to obtain a solution of the form $\zeta(t)w$ of the n-body problem we must solve the complex differential equation $\ddot{\zeta}\zeta^{\frac{1}{2}}\bar{\zeta}^{\frac{3}{2}} = -\lambda$ and we must have a solution to (2.1). The solutions to the differential equation are the parameterized conics in the plane familiar from the study of the two-body problem. For a good treatment of the two-body problem see [Pl]. In particular, one solution is $\zeta = e^{i\sqrt{\lambda}t}$ which corresponds to every particle having a circular orbit. If such a system were viewed in rotating coordinates with angular speed $\omega = \sqrt{\lambda}$ it would be an equilibrium, which is why planar central configurations are often referred to as *relative equilibria*. In conclusion, every homographic solution of the planar n-body problem is generated by a n-body central configuration in which each point moves on a Keplerian orbit.

Central configurations are ubiquitous in the study of the n-body problem. Some of the more important problems in which they arise will be summarized.

It is natural to inquire about the collisions of some subset of particles in the n-body problem, as this introduces obvious singularities one might wish to either study or avoid. Central configurations turn out to be the limit configurations in collisions, a property that holds for a range of exponents for forces which are powers of interparticle distances, not only that which is derived from the Newtonian potential [Sa1]. More precisely, in the Newtonian case, if n points in the n-body problem collide simultaneously at a finite time t^* then the rescaled position vector $q^* = (t - t^*)^{-2/3}q$ has as its limit a central configuration with the same mass vector [Su], [Wn].

Perhaps more surprisingly, a similar phenomenon occurs for expanding gravitational systems:

Theorem 2 (Saari [Sa2])**.** *If the total energy $E = \sum m_i |\dot{q}_i|^2 + U$ of a solution of the n-body problem is positive and the solution exists for $t \in [0, \infty)$ then either*

$$(max_{i,k}\{|q_i(t) - q_k(t)|\})\,/t \to \infty \text{ and } min_{i \neq k}\{|q_i(t) - q_k(t)|\} \to 0$$

$$or$$

$$\text{for every } i \text{ there exists an } A_i \in \mathbb{R} \text{ such that } |q_i| = A_i t + O(t^{2/3}).$$

The first case in Theorem 2 is a pathology that is an interesting topic in itself [Pa], [DH], [SX], but which is not closely related to central configurations. In the second case, if two particles have the same constant in the linear term we say they are part of a subsystem. Each subsystem can then be further divided into groups of masses whose centers of mass limit to central configurations around the subsystem's center of mass. Similar results hold even in the zero energy case [SH].

Another interesting question in which central configurations arise, introduced by Poincaré [Po2] and Birkhoff [B], is the determination of the topology of the n-body problem, i.e. the cohomology of the common level sets of the energy and angular momentum functions which are conserved along trajectories. These level sets are often called the integral manifolds, although they are not always manifolds in the strict sense. Central configurations occur in this context as critical points of the functions defining the integral manifolds [Sm1], [C], [Ea], [Al1]. One hope in studying the topology of the integral manifolds is that one can obtain some information about the dynamics. A nice example of this is in the two-body problem, in which the topology and dynamics were beautifully combined by Moser [Ms1] in the result that the integral manifolds are diffeomorphic to T_0S^2, the unit tangent bundle of S^2, and the trajectories are given by geodesic flow. Recently the cohomology of the integral manifolds in the planar and spatial three-body problem was completely determined [MMW]. A much better understanding of the four-body central configurations will be necessary before such investigations are begun for the four-body problem.

Finally, and possibly most interesting from a mathematical viewpoint, central configurations are the only real landmarks in the study of the dynamics of the n-body problem. Almost every result in three-body dynamics is related to central configurations. For many of these applications it is crucial to know about the stability of the configurations to perturbations, which amounts to understanding the linearization of the dynamics about the central configurations. Perhaps the most striking example of the use of central configurations in this context is the work of Moeckel on chaotic dynamics in the three-body problem [Mo2] using blow-up techniques introduced by McGehee [Mc], [Ms2].

Chapter 3

CLASSIFICATION OF CENTRAL CONFIGURATIONS

Some of the historically and mathematically interesting results on central configurations will be summarized in this chapter, including the three-body case and a relatively recent theorem that applies for any n, the perpendicular bisector theorem. Finally, I present two corollaries of mine to the perpendicular bisector theorem.

Relative equilibria are well understood in the case $n = 3$ (the $n = 2$ case is trivial, as all line segments are similar). Euler and Lagrange solved this case in the 18th century [Eu], [La]. For a given mass vector there are five relative equilibria (up to rotation, dilation, and translation equivalence). The masses can be arranged in an equilateral triangle, in two different orders, or they can be collinear in any order. Note the surprising fact that the equilateral triangle is a central configuration for any choice of positive masses. The spacing in the collinear cases is given by a polynomial:

$$(m_1 + m_3)p^5 - (2m_1 + 3m_3)p^4 + (m_1 + 2m_2 + 3m_3)p^3$$
$$-(m_1 + 3m_2)p^2 + (2m_1 + 3m_2)p - (m_1 + m_2) = 0$$

where p is the ratio of the distance between the first two points and the distance between the first and third points, for the case in which the second point is in the middle. These results can be obtained by relatively straightforward algebra [SM].

The classification problem for the general collinear case ($d = 1$) was solved by Moulton [Mu] in 1910. He proved that there is a unique equivalence class of central configurations for any ordering of n masses on a line. If we consider the collinear problem as embedded in a higher dimensional space, $d > 1$, than for any mass vector $m \in \mathbb{R}^n_+$ there are $n!/2$ affine equivalence classes of collinear central configurations. A more modern proof of this result is contained in a paper of Smale [Sm1].

Cases in which some of the masses are infinitesimal have been studied by several authors [Hi], [Pe1], [Ho], [G], [Ar], [Ll], and [Mo5]. In particular the results of Xia [X] provide strong lower bounds on the number of relative equilibria with certain types of mass vectors. These methods typically rely on knowledge of the two- and three-body cases, and could be extended if the four-body case were better understood.

For $n > 4$, there are many results about highly symmetric configurations, such as nested regular polygons or regular polygons with a central mass [Hp], [L-F], [Lo], [An], [Br], [Li], [Kl], [PW], [El], [Ce], [Bu2], [CLN], [F], [Gl], [ZZ]. Equal masses arranged in a regular polygon with an arbitrary central variable mass are always central configurations; their stability under perturbations is a difficult topic that has been extensively studied [Ra], [Mo4], [Ro1]. The historical interest in this subject is partly due to the work of Maxwell on the rings of Saturn [Ma].

A theorem of Conley and Moeckel, the perpendicular bisector theorem, is useful in restricting the possible geometries of central configurations. This theorem actually applies for any n, in $\mathbb{R}^3$ or $\mathbb{R}^2$, but it is most useful for small values of n ($n = 3$, 4, and 5). For clarity the theorem will first be stated in the planar case. Note that any two intersecting lines in the plane determine a pair of open double cones.

Theorem 3 (Perpendicular bisector theorem [Mo3])**.**
If q_i and q_j are any two points of a planar central configuration, then the pair of double cones determined by the perpendicular bisector of q_i and q_j and the line through q_i and q_j must either contain points of the central configuration in each double cone, or have no points in either of the double cones.

In the spatial case, the line through q_i and q_j is replaced by any plane through q_i and q_j, and the perpendicular bisector is likewise interpreted as the plane perpendicular to the the line through q_i and q_j which contains the midpoint of q_i and q_j. Once the planes are chosen they determine two three-dimensional double cones, and with these modifications the theorem still holds.

As a simple example to illustrate the above theorem, take $q_i = (-1, 0)$ and $q_j = (1, 0)$ so the two relevant lines would simply be the coordinate axes. Then one cone would consist of the first and third quadrants and the other cone would be the second and fourth quadrants. In this case, if there were any points in the interior of the first or third quadrants then there would have to be a point in the interior of the second or fourth quadrants.

In the $n = 3$ case the perpendicular bisector theorem almost immediately implies Lagrange's result that the only non-collinear central configurations are equilateral triangles. To see this note that for each pair of points the theorem says that the third point must lie on the perpendicular bisector of the pair. This means that the distances from each member of the pair to the third point are equal, and since this is true for every pair all the sides are equal in length.

One corollary to Theorem 3 is that it is impossible to have an n-body central configuration with exactly $n-1$ collinear points, for $n > 3$. The non-collinear point would have to be on all the perpendicular bisectors of the collinear points, which is impossible for $n > 3$ since the bisectors are distinct parallel lines. This result was first found by a more complicated argument in 1988 [ZY].

Planar four-body central configurations which are not collinear can be classified as either *convex* or *concave*. A concave configuration has one point which is strictly inside the convex hull of the other three points. A convex configuration does not have a point contained in the convex hull of the other three points. Another corollary of mine to the perpendicular bisector theorem is that a concave relative equilibrium for the $n = 4$ case cannot have an oblique triangle as its convex hull (i.e. the triangle formed by the three outer masses). This is easily seen by considering the pairs of masses that would form the two shortest sides of the outer triangle, and the fact that the intersection of the perpendicular bisectors of these two pairs intersect outside the triangle.

Just as the perpendicular bisector theorem restricts the $n = 3$ non-collinear configurations to just one possibility, the equilateral triangle, it also eliminates all non-planar four-body central configurations except the regular tetrahedron. We will see in Chapter 6 that Theorem 3 is useful in the planar four-body case; the corresponding application of the perpendicular bisector theorem to the spatial five-body case is an intriguing area that a paper of Schmidt [Sc] begins to explore.

Chapter 4

FOUR-BODY CENTRAL CONFIGURATIONS

The four-body classification of central configurations is much more difficult than the three-body case, and in general is unsolved. In the three-body case, the equilateral triangle is a relative equilibrium for any choice of masses. We will see below that in the $n = 4$ planar problem there are no configurations which are central configurations for every mass vector, although it is true that any n masses arranged in a regular $n-1$-simplex in $\mathbb{R}^{n-1}$ form a central configuration [Sa1]. Note that Lagrange's configuration is a special case of Saari's result with $n = 3$; the $n = 4$ case was actually first found by Lehmann-Filhes in 1891 [L-F]. There are two special cases in which the four-body central configurations are understood. The limiting case in which one of the four masses goes to zero, often referred to as the $3 + 1$ case, has been studied by Gannaway [G] and Arensdorf [Ar], and the case in which all four masses are equal has been recently classified by Albouy [Al2]. For the equal mass case there are only four equivalence classes of planar central configurations: the square, an equilateral triangle with a mass at its center, a collinear configuration, and a particular isosceles triangle with another mass on its axis of symmetry. This last type of central configuration, with two distinct pairs of equal interparticle distances, will be referred to as an *isosceles configuration*. Note that for the equal mass case all the central configurations possess at least one axis of symmetry [Al1] under reflections. For comparison with the generic case it can be more helpful to count the affine equivalence classes; there are 50 affine equivalence classes of planar central configurations for the equal mass case.

It has been shown [Ku], [Mo1] that the number of four-body central configurations with given masses is generically finite, but there may be an at most codimension-1 subset of the space of masses where the number of central configurations is infinite. Recall that in counting central configurations we actually count their equivalence classes as described in Chapter 2. This question of the finiteness of the number of central configurations is listed by Smale as one of his "Mathematical problems for the next century" [Sm2]. The generic upper bound given by Moeckel is so large ($39 \cdot 77^{14}$) that there can be little doubt that improvements are possible. I conjecture that in fact the number of 4-body central configurations is finite for any given mass vector. Moeckel also defines the *bifurcation set* $\mathcal{B}$ in the space of masses to be the complement of the set with the property that in some neighborhood of each point the number of central configurations is constant, i.e. the set $\mathcal{B}$ is where the number of central configurations changes. Little is known about the structure of $\mathcal{B}$, although some numerical investigations have been done [Si], [G]. Very recently Moeckel proved a more general finiteness result that applies to any n bodies in $\mathbb{R}^{n-2}$ [Mo7]. It also worth noting in this context that if the assumption of positive masses is eliminated there is an example of a continuum of central configurations in the five-body problem [Ro2].

The article of MacMillan and Bartky [MB] contains some of the background and tools relevant to research on the existence of concave relative equilibria in the four-body problem. Some of their results will be summarized here. The exposition given here will be partially based on later treatments of the same material, in particular a paper by Williams [Wi] which generalized some of the techniques to the planar five-body problem. Some of [MB] was also nicely summarized in [Sc], who extended their techniques to the spatial five-body problem.

The first step is to recast the equations defining a planar central configuration, so that the variables are the distances between the particles rather than their coordinates. This not only reduces the number of variables but also eliminates the rotational and translational degeneracy of the equivalence classes of central configurations, leaving only the dilational degeneracy. Dziobek was the first to introduce this approach [D] and exploit its significant advantages.

Assume that the points of the configuration are not collinear, as the theorem of Moulton [Mu] mentioned previously covers that case. Consider (2.1) for the coordinates of the first point of the central configuration, where $q_i = (x_i, y_i)$ and $r_{ij} = |q_i - q_j|^3$:

$$\lambda x_1 = \sum_{j \neq 1} m_j \frac{x_1 - x_j}{r_{1j}^3}$$

$$\lambda y_1 = \sum_{j\neq 1} m_j \frac{y_1 - y_j}{r_{1j}^3}.$$

The first point's equations were chosen for simplicity, but the following manipulations can be done, *mutatis mutandis*, for the others.

Now we will change from using the arbitrary parameter λ to r_0 defined by $\lambda = M r_0^{-3}$; recall that $M = \sum_{i=1}^n m_i$. For convenience we will also define $R_0 = r_0^{-3}$, $R_{ij} = r_{ij}^{-3}$. Then our equations become:

$$\sum_{j\neq 1} (R_{1j} - R_0)(x_1 - x_j) m_j = 0$$
$$\sum_{j\neq 1} (R_{1j} - R_0)(y_1 - y_j) m_j = 0.$$

To eliminate the m_2 term from these equations, we simply cross-multiply by the respective coefficients of that term and subtract the two equations. Our result is

$$-2(R_{13} - R_0)\Delta_4 m_3 + 2(R_{14} - R_0)\Delta_3 m_4 = 0$$

where Δ_i is the signed area of the triangle formed by the q_j for which $j \neq i$. We will fix the signs by the convention that if the points of the triangle are listed in counterclockwise order and the missing point appended to the end of the list, then the sign of the permutation taking (1234) to the list is the sign of Δ_i. For example, if the four points form a square, with the points in counterclockwise ascending order, then $\Delta_1 = \Delta_3 < 0$ and $\Delta_2 = \Delta_4 > 0$. An interesting property of this sign convention is that $\sum_{i=1}^4 \Delta_i = 0$. In the non-collinear case we know that no three of the masses can be collinear, by one of the corollaries to the perpendicular bisector theorem discussed in Chapter 3, so none of the Δ_i are zero. The absolute value of Δ_i will be denoted by D_i.

Similarly, from the equations for q_2 we can eliminate the m_1 terms and obtain

$$-2(R_{23} - R_0)\Delta_4 m_3 + 2(R_{24} - R_0)\Delta_3 m_4 = 0. \tag{4.1}$$

If we introduce the notation $S_{ij} = R_{ij} - R_0$ then the above pair of equations implies that

$$S_{13}S_{24} = S_{23}S_{14}.$$

This is the stage where the collinear case must be excluded, since the Δ_i would be 0 for every $i = 1, \ldots, 4$. Also notice that if any of the $S_{ij} = 0$ then the equations obtained by permuting the indices of (4.1) imply that all of the S_{ij} are zero, and this is impossible since four planar points cannot all be the same distance from one another.

Applying the same argument to the rest of the equations yields only one more independent equation, so that we have what we will call the *consistency equations*:

$$S_{13}S_{24} = S_{23}S_{14} = S_{12}S_{34}. \tag{4.2}$$

These are extremely useful necessary conditions for a four-body central configuration; they are also sufficient except for the positivity of the masses, which we can check using the formulae for the mass ratios that follow immediately from the above argument:

$$\frac{m_i}{m_j} = \frac{\Delta_i S_{jk}}{\Delta_j S_{ik}} \tag{4.3}$$

where i, j, and k are distinct from one another. Note that the mass ratios are uniquely determined by the distances and the parameter R_0. In all but one case the R_0 parameter is determined by the consistency equations, so that the masses can be determined from the interparticle distances. The exception is the highly symmetric configuration consisting of an equilateral triangle with the fourth mass at its center. We will refer to this symmetrical configuration as the equilateral configuration. In the other cases we can explicitly solve for R_0 from the consistency equations by some elementary algebra:

$$R_0 = \frac{R_{12}R_{34} - R_{23}R_{14}}{R_{12} + R_{34} - R_{23} - R_{14}} =$$
$$\frac{R_{23}R_{14} - R_{13}R_{24}}{R_{23} + R_{14} - R_{13} - R_{24}} = \frac{R_{13}R_{24} - R_{12}R_{34}}{R_{13} + R_{24} - R_{12} - R_{34}} \; ,$$

provided that the denominators in the above equations are nonzero. Since we are assuming that R_0 exists and is finite, if one of the above denominators vanishes then the numerator must be zero as well. A little algebra then shows that there are two pairs of equal adjacent sides, e.g. if the first equation above was indeterminate then either $r_{12} = r_{23}$ and $r_{34} = r_{14}$, or $r_{12} = r_{14}$ and $r_{34} = r_{23}$. In either case the configuration is an isosceles configuration. In this case one of the other two expressions for R_0 must be used. If all of the above equations are indeterminate, then there must be two distinct triplets of equal adjacent sides, which is only possible for the equilateral configuration.

For the non-equilateral configurations, if the parameter R_0 is eliminated the equations for the mass ratios simplify to:

$$\frac{m_i}{m_j} = \frac{\Delta_i(R_{jk} - R_{jl})}{\Delta_j(R_{ik} - R_{il})}. \tag{4.4}$$

These equations for the mass ratios were not presented by MacMillan and Bartky, but appear in [Sc].

For the equilateral configuration the outer masses must be equal and the ratio of the inner mass to the outer masses can be any positive number, which can be checked directly from (4.3). Note that if we set the total mass $M = \sum_{i=1}^{4} m_i = 1$ then the masses are uniquely determined by the mass ratios. The set $\mathcal{M} = \{m \in \mathbb{R}^4 \mid \sum_{i=1}^{4} m_i = 1,\ m_i \geq 0\}$ is a tetrahedron, which will be referred to as the *mass tetrahedron*. The segment of the mass tetrahedron given by $m_1 = m_2 = m_3$ will be referred to as the *mass core*; the fourth mass has been singled out arbitrarily. Let us also denote the set of affine equivalence classes of non-collinear planar four-body central configurations by $\mathcal{P}$.

In summary, we have obtained the following:

Theorem 4. *Every element of $\mathcal{P}\backslash\{$Equilateral configuration$\}$ has a unique mass vector in the interior of $\mathcal{M}$ which makes the configuration a central configuration. For the equilateral configuration the outer three masses must be equal.*

The fact that the equilateral configuration, with the fourth mass in the middle, can have masses anywhere on the mass core leads to various questions, such as what other central configurations have masses on this segment. Recall from Chapter 2 Dziobek's description of central configurations as critical points of a gradient flow on the space $\tilde{S} = S/SO(d)$, where both S and the flow depend on the masses. It has been shown [Pm] that the critical point corresponding to the equilateral configuration is nondegenerate for any masses on the mass core except at one point, where $m_4/m_1 = (81 + 64\sqrt{3})/249$. By nondegenerate we mean that the Hessian of the function $U|_{\tilde{S}}$ determining the gradient flow is nondegenerate. At this special point on the mass core another family of configurations with masses on the mass core bifurcates from the equilateral configuration. This new family consists of isosceles configurations [MS]. The isosceles family is only known to exist locally, near the particular mass value mentioned above. However, it seems possible that the masses of this family include the entire mass core. An intriguing piece of evidence in this regard is the isosceles configuration discovered by Albouy, which has four equal masses and is presumably a member of the above family.

Although using the interparticle distances as variables has many advantages, there is a serious disadvantage to their use in the planar four-body problem, namely that they are not independent. There is one relation between them, which corresponds to the geometric fact that the volume V of the tetrahedron formed by the four points must vanish. This relation is conveniently expressed by the vanishing of the following determinant [Hg], the value of which we shall denote by P:

$$P(r_{12}, r_{23}, r_{34}, r_{14}, r_{13}, r_{24}) = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 \\ 1 & r_{12}^2 & 0 & r_{23}^2 & r_{24}^2 \\ 1 & r_{13}^2 & r_{23}^2 & 0 & r_{34}^2 \\ 1 & r_{14}^2 & r_{24}^2 & r_{34}^2 & 0 \end{vmatrix} = 288V^2. \tag{4.5}$$

The above formula for the volume of a tetrahedron, which is the three-dimensional analogue of Heron's formula, dates from at least the 19th century. Analogous formulae give the volumes of simplices in any dimension; the determinants are known as Cayley-Menger determinants.

Chapter 5

CONVEX CENTRAL CONFIGURATIONS

Convex central configurations in the four-body problem are better understood than the concave case. It is known that for any set of positive masses there is at least one convex relative equilibrium [MB]. The available evidence suggests that the concave configurations have a more complicated structure than the convex configurations. Because of its relevance, the convex case will be summarized here in some detail, closely following the original argument.

A more precise statement of the result in [MB] is:

Theorem 5. *Given* $k_0, k_{12}, k_{34}, k_{13} > 0$*, there exists a convex central configuration such that* $k_0 = r_0$*,* $max(r_{12}, r_{23}, r_{34}, r_{14}) < r_0 < min(r_{13}, r_{24})$*, and the mass ratios are given by* $\frac{m_1}{m_2} = k_{12}$*,* $\frac{m_3}{m_4} = k_{34}$*,* $\frac{m_1}{m_3} = k_{13}$*.*

Proof. We will restrict attention in the proof to the convex configurations which are ordered counter-clockwise as q_1, q_2, q_3, q_4. By examining the mass equations in this case one can determine that for all the masses to be positive the inequalities $\max(r_{12}, r_{23}, r_{34}, r_{14}) < r_0 < \min(r_{13}, r_{24})$ must be satisfied. Since we will immediately set $r_0 = k_0$ this puts some restrictions on the interparticle distances. Note that specifying the value of r_0 merely removes the dilational degeneracy.

Consider the situation for fixed $r_{12} < r_0$, represented in the figure below.

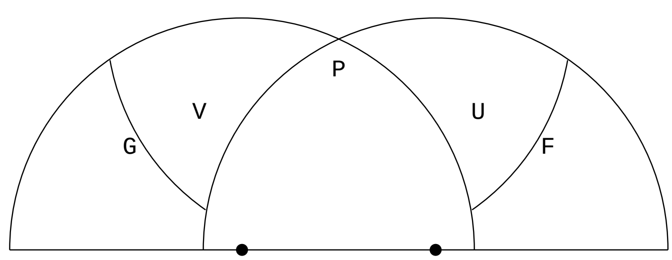


Figure 5.1: Starting point of convex construction

The first and second points of the configuration have been rotated to lie horizontally, and a semicircle of radius r_0 drawn around each of them. The first point is the point on the left. The two other arcs F and G in the figure are a distance r_0 from the the intersection point P. The region U is defined as the intersection of the interiors of the circles of radius r_0 around the second point and the point P and the exterior of the circle of radius r_0 around the first point. The region V is defined similarly with the roles of the first and second points interchanged.

We will proceed to prove that for every point in the region U there is one point in the region V such that the four points form a central configuration for some mass vector. In order for r_0 to satisfy the inequalities in the above theorem we must restrict the fourth point to be in W, the intersection of V and the interior of the circle centered at the third point with radius r_0. We will show that given the position of the third point, in U, we can find a fourth point, in W, such that $S_{13}S_{24} = S_{23}S_{14} = S_{12}S_{34}$. Recall that $S_{ij} = R_{ij} - R_0$. Having fixed our first three points, we are given R_0, R_{12}, R_{23}, and R_{13}.

Let ρ_0 denote the distance from the third point to P. Now fix a positive ρ such that $\rho_0 < \rho < r_0$ and consider the arc AB in W which is a distance ρ from the third point. One of the consistency equations can be written as $(R_{23} - R_0)(R_{14} - R_0) = (R_0 - R_{13})(R_0 - R_{24})$ in which all the factors in parentheses are positive if the fourth point lies on the interior of the arc AB. At point A, $R_{24} = R_0$ and thus the function $(R_0 - R_{13})(R_0 - R_{24})$ is zero. As the fourth point is moved up the arc AB towards B, this function monotonically increases since the first factor is constant. Likewise, if we consider the function $(R_{23} - R_0)(R_{14} - R_0)$ we see that it is zero when the fourth point is at B and increases monotonically as the fourth point is moved along the arc AB towards A.

So the above consistency equation can be satisfied for any such ρ by a unique fourth point on the arc AB. Define $\mu(\rho) = (R_{23}-R_0)(R_{14}-R_0) = (R_0-R_{13})(R_0-R_{24})$ to be the common value of these two functions at such a point. We know that $\mu(\rho_0) = 0$ by its definition. Also $\frac{d\mu}{d\rho} > 0$ when $\rho = \rho_0$; to see this, first note that $\frac{d\mu}{d\rho} = (R_{23} - R_0)\frac{dR_{14}}{d\rho} = -(R_0 - R_{13})\frac{dR_{24}}{d\rho}$ and either $R_{24} = r_{24}^{-3}$ must decrease or R_{14} must increase as a function of ρ. By linearizing about $r_{14} = r_{24} = r_0$, it can be shown that at least one of these changes is first order in ρ. Since r_{24} and r_{14} can be constant to first order under a perturbation of the fourth

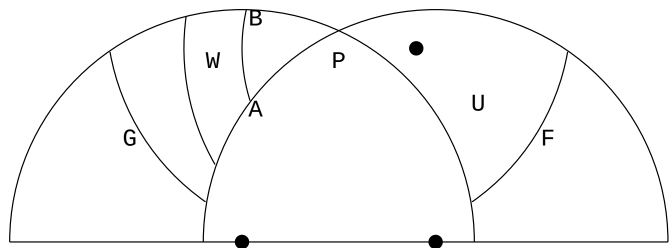


Figure 5.2: Convex construction

point only along the line joining the first and second points, which does not intersect W, we see that $\frac{d\mu}{d\rho}$ cannot be zero in W and thus must remain positive along the curve of fourth points satisfying $S_{13}S_{24} = S_{23}S_{14}$. The function $S_{12}S_{34} = (R_{12} - R_0)(R_{34} - R_0)$ decreases as $\rho = r_{34}$ increases, and vanishes on the arc where $r_{34} = r_0$, so there is a unique point in W where the consistency equations hold.

We now examine the dependence of the masses in the convex case as described above. Consider any curve CD of positions of the third mass in U that begins from the arc of radius r_0 around the first point and ends on the arc of radius r_0 around the second point. For each point on this curve, we have seen that there is a unique fourth point in V that is a four-body relative equilibrium. It is clear from the geometry that for any such 1-parameter family of relative equilibria the areal ratios $|\Delta_i/\Delta_j|$ will be bounded above and below by some positive numbers since no three of the masses can get arbitrarily close to being collinear. Examining (4.3) with $i = 1, j = 2$ we see that the ratio will become infinite at C and zero at D. Thus there must be at least one point on the curve with $m_1/m_2 = k_{12}$. Since the masses are continuous functions of the distances in U, by varying the curve CD we obtain at least one arc C_1 in U, beginning at P and ending on F, such that the central configuration obtained by placing the third point on the curve has mass ratio $m_1/m_2 = k_{12}$.

For each point on C_1 there is a unique corresponding point in V for the fourth point, and thus there is a corresponding curve in V to C_1. By construction, as the third point's location on C_1 approaches F, the corresponding point in V must approach P. By examining (4.3) and noting that S_{14} is limiting to zero in this case we see the ratio m_3/m_4 goes to zero. From the symmetry of the argument, as the third point approaches P, the ratio m_3/m_4 must go to infinity. This implies that there is at least one point on C_1 such that $m_3/m_4 = k_{34}$.

So far it has been shown that for $r_{12} < r_0$, both fixed, there exists a central configuration with $m_1/m_2 = k_{12}$ and $m_3/m_4 = k_{34}$. Consider a family of such configurations as r_{12} is decreased to zero while r_0 is held fixed. This means the limit of r_{13}/r_{23} and r_{14}/r_{24} is 1, which means that in fact each of the distances in these ratios must tend to r_0 by the inequalities $r_{23} < r_0 < r_{13}$ and $r_{14} < r_0 < r_{24}$. This means that right side of the equation $S_{12}S_{34} = S_{23}S_{14}$ is limiting to zero, but the S_{12} term is getting arbitrarily large. This forces r_{34} to also limit to r_0. From (4.3) we can obtain $\frac{m_1 m_2}{m_3 m_4} = \frac{\Delta_1 \Delta_2 S_{34}}{\Delta_3 \Delta_4 S_{12}}$. Since S_{34}/S_{12} is going to zero while the other ratios are bounded, the ratio $m_1 m_2/m_3 m_4$ limits to zero along with r_{12}.

Similarly, if we consider what happens when r_{12} is increased to r_0, we find that the ratios of areas in the mass equations remain bounded and S_{12} goes to zero while S_{34} is bounded away from zero. So $m_1 m_2/m_3 m_4$ becomes arbitrarily large and there is at least one $r_{12} < r_0$ such that $m_1 m_2/m_3 m_4 = k_{13}^2 k_{34}/k_{12}$, which in turn implies that $m_1/m_3 = k_{13}$ as desired. This completes the proof of Theorem 5. $\square$

Theorem 5 may be summarized by saying that the map from the interparticle distances of the convex configurations into the interior of the mass tetrahedron $\mathcal{M}^\circ$ is surjective. This mapping is well defined by Theorem 4, since the equilateral configuration is not convex. Note that the proof was essentially topological, relying on knowledge of the behavior of the masses at the boundaries of the possible convex configurations to infer the surjectivity of the map on the interior.

Chapter 6

CANONICAL CONCAVE CONFIGURATIONS

The following theorem gives a canonical way to chose the representatives of equivalence classes of concave configurations. A configuration is called concave if it has one of its points strictly inside the convex hull of the other three points. The notation θ_{ijk} will denote the angle between the segments $\overline{ij}$ and $\overline{jk}$.

Theorem 6. *Every equivalence class of concave planar four-body central configurations has a representative with outer points* $1, 2, 3$ *such that* $r_{12} = 1 \leq r_{23} \leq r_{13}$ *and the angles of the outer triangle satisfy* $\theta_{123} < 90°$*,* $\theta_{132} > 30°$*.*

Proof. It will be assumed that r_{12} is one of the shortest of the outer sides of the triangle. In order to deal with the rotational and dilational symmetries of equivalence classes of configurations, the line segment from q_1 to q_2 will be assumed horizontal, length one, and with q_2 to the right of q_1. Also, the third point will be assumed to be above the segment $q_1 q_2$, and to the right of the perpendicular bisector of q_1 and q_2. The fourth point q_4 will be in the convex hull of the first three; again there is no loss of generality here since we are free to reorder the points. Let Q denote the region to the right of and including the perpendicular bisector of q_1 and q_2, to the left of the vertical line through the second point, outside or on the circle of radius one around the second point, and inside the circle of radius one around the point e which is at unit distance from q_1 and q_2. However, the point e is not included in Q. Figure 6.1 shows the region Q.

For a triangle of side lengths l_1, l_2, and l_3, and area A, the circumcenter is at a distance from each vertex of $r_C = \frac{l_1 l_2 l_3}{4A}$, known as the circumradius. From our symmetry-breaking assumptions and the requirement that the masses are positive one can determine from the mass equations that [MB]

$$r_{34} \leq r_{14} \leq r_{24} \leq r_0 \leq r_{12} = 1 \leq r_{23} \leq r_{13}. \tag{6.1}$$

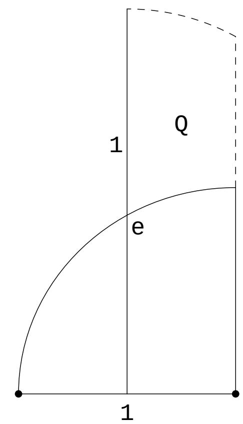


Figure 6.1: Region of possible positions of the third point

Once the position of the third mass is chosen, the above inequalities and the perpendicular bisector theorem restrict the position of the fourth point. There are three possibilities for the allowable positions of the fourth mass, depending on whether the third point is above or below the diagonal defined by $r_{23}^2 = r_{13}^2 - r_{13} + 1$ in region Q. Two of these are illustrated in Figure 6.2. If the third point is above the diagonal, let V be the region bounded by the perpendicular bisectors of the segments $\overline{12}$ and $\overline{13}$, the segment $\overline{13}$, and the arc of radius r_{12} around the second point. In some cases the segment $\overline{13}$ is not part of the boundary (this is the case not shown in Figure 6.2). If the third point is below the diagonal, then let V be the region bounded by the segment $\overline{13}$ and the two perpendicular bisectors of the segments $\overline{12}$ and $\overline{13}$. It can be shown by the perpendicular bisector theorem that the fourth point must be contained in V.

Note that the circle of radius one around the point e consists of points for which $\theta_{132} = 30°$, by an elementary theorem in geometry. On this boundary arc of Q, and on the vertical line through the second point, the region V consists only of the circumcenter point. □

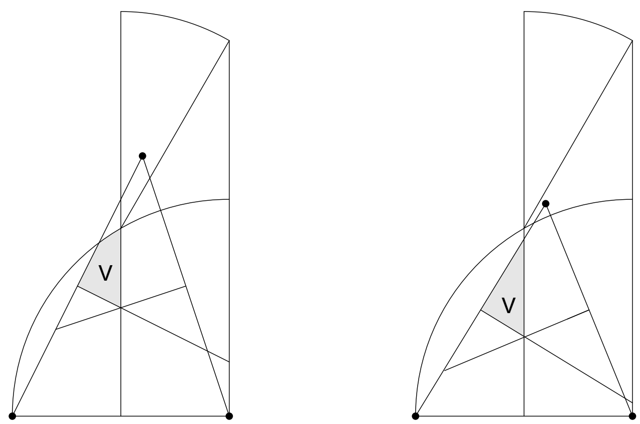


Figure 6.2: Regions of possible positions for the interior point

Chapter 7

AN EXISTENCE THEOREM FOR CONCAVE CENTRAL CONFIGURATIONS

In this chapter some facts about the existence of concave central configurations are proved. While they have some interest in their own right, they are also necessary ingredients for later results.

With the conventions of the preceding chapter, we begin by studying points in the region Q, which gives us the inequalities $r_{12} = 1 \le r_{23} \le r_{13} < 2$ and $\theta_{123} < 90°$, $\theta_{132} > 30°$. These conditions on θ_{123} and θ_{132}, when combined with the condition that $r_{23} \le r_{13}$, are equivalent to:

$$1 < r_{13} < 2 \text{ and}$$

$$\begin{cases} 1 \le r_{23} \le r_{13} & \text{if } 1 < r_{13} \le \sqrt{2} \\ \sqrt{r_{13}^2 - 1} < r_{23} \le r_{13} & \text{if } \sqrt{2} \le r_{13} \le \sqrt{2+\sqrt{3}} \\ \sqrt{r_{13}^2 - 1} < r_{23} < \frac{\sqrt{3}}{2} r_{13} + \sqrt{4 - r_{13}^2}/2 & \text{if } \sqrt{2+\sqrt{3}} \le r_{13} < 2. \end{cases} \tag{7.1}$$

Definition 2. Labeled triangles with the third point in Q will be called strictly admissible triangles, and those with the third point in $\bar{Q}$ will be called admissible.

In particular, the side lengths of any admissible triangle satisfy the inequalities in (7.1). For such configurations we have an existence theorem. For convenience let us use the notation $R_C = r_C^{-3}$ for the inverse cube of the circumradius. For admissible triangles the circumradius is in the range $[1/\sqrt{3}, 1]$.

The following theorem gives us much of the existence result we desire, although it sidesteps the thorny issue of concavity.

Theorem 7. *For a strictly admissible triangle, and any R_0 satisfying $1 < R_0 < R_C$, there exists a fourth point for which $r_{34} \leq r_{14} \leq r_{24} < r_0$ and such that the four points satisfy the consistency equations (4.2).*

Proof. The strategy of the proof is to first show that there are positive numbers $\{r_{12}, r_{23}, z_{34}, z_{14}, r_{13}, z_{24}\}$ satisfying the consistency equations and the planarity condition $P = 0$. Then these numbers will be shown to be interparticle distances of a planar configuration of four points.

First we pick the points of the outer triangle (points 1, 2, and 3) so that (7.1) are satisfied, and we pick a value of the parameter R_0 satisfying $1 < R_0 < R_C$.

Note that the consistency equations (4.2) are equivalent to

$$R_{14} = F_{14}(r_{24}, r_0) \tag{7.2}$$

and

$$R_{34} = F_{34}(r_{24}, r_0). \tag{7.3}$$

where

$$F_{14}(r_{24}, r_0) \equiv R_0 + \frac{S_{13}}{S_{23}}(R_{24} - R_0) \tag{7.4}$$

and

$$F_{34}(r_{24}, r_0) \equiv R_0 + \frac{S_{13}}{S_{12}}(R_{24} - R_0). \tag{7.5}$$

In defining the above functions $F_{14}(r_{24}, r_0)$ and $F_{34}(r_{24}, r_0)$ we are suppressing their dependence on r_{13} and r_{23} since these are fixed parameters throughout the argument. We will also use the notation $f_{34} = F_{34}^{-\frac{1}{3}}$ and $f_{14} = F_{14}^{-\frac{1}{3}}$.

A little algebra together with our assumed inequalities shows that $1 \leq \frac{S_{13}}{S_{23}} \leq \frac{S_{13}}{S_{12}}$; at least one of these inequalities must be strict. We will consider the the interval of R_{24} values $R_0 \leq R_{24} \leq R_C$. In this interval, $F_{34} \geq F_{14} \geq R_{24}$ by definition, with both equalities holding only at the endpoint $R_{24} = R_0$.

We will first show that $P(1, r_{23}, f_{34}, f_{14}, r_{13}, r_{24}) = 0$ for a value of $R_{24} \in (R_0, R_C)$. At one endpoint of the interval, $R_{24} = F_{34} = F_{14} = R_0 < R_C$. For these values the function

$P(1, r_{23}, r_0, r_0, r_{13}, r_0)$ can be simplified to:

$$2r_0^2(2r_{23}^2 + 2r_{13}^2 + 2r_{23}^2 r_{13}^2 - 1 - r_{13}^4 - r_{23}^4) - 2r_{23}^2 r_{13}^2 = 32r_0^2 D_4^2 - 2r_{23}^2 r_{13}^2.$$

Recall that D_4 is the area of the triangle not containing the fourth point, and that the circumradius r_C of this triangle is $\frac{r_{23}r_{13}}{4D_4}$. Then it is easy to see that $P(1, r_{23}, r_C, r_C, r_{13}, r_C) = 0$. Furthermore, since $D_4^2 > 0$ we see that for $r_0 > r_C$ the value of P must increase, i.e. $P(1, r_{23}, r_0, r_0, r_{13}, r_0) > 0$ for $r_0 > r_C$.

So $P > 0$ at one endpoint, and we must show it is negative at the other endpoint, where $F_{34} \geq F_{14} \geq R_{24} = R_C$ (with at least one strict inequality). For this we need the following lemma:

Lemma 1. *For $r_{34} \leq r_{14} \leq r_{24} \leq r_{12} = 1 \leq r_{23} \leq r_{13}$, $\partial P/\partial r_{14} > 0$ and $\partial P/\partial r_{34} > 0$.*

Proof. We simply explicitly compute the derivatives and regroup the terms as below:

$$\partial P/\partial r_{14} = 4r_{14}((r_{13}^2 - 1)(r_{24}^2 - r_{34}^2) + r_{23}^2(r_{34}^2 + r_{13}^2 - r_{23}^2 + r_{24}^2 + 1 - 2r_{14}^2))$$
$$\partial P/\partial r_{34} = 4r_{34}((r_{13}^2 - r_{23}^2)(r_{24}^2 - r_{14}^2) + (r_{14}^2 + r_{13}^2 - 1 + r_{24}^2 + r_{23}^2 - 2r_{34}^2)).$$

Our assumptions imply that every expression in parentheses in the above formulae is non-negative, and at least one term must be positive, so the partial derivatives are positive. □

At the endpoint we are considering, only the third and fourth arguments could have changed from the arguments in the equation $P(1, r_{23}, r_C, r_C, r_{13}, r_C) = 0$, and at least one of them decreased, so $P(1, r_{23}, f_{34}, f_{14}, r_{13}, r_C) < 0$ at this endpoint.

Since $P(1, r_{23}, f_{34}, f_{14}, r_{13}, r_{24})$ is a smooth function of r_{24} for fixed r_0, the intermediate value theorem ensures that there is at least one zero of $P(1, r_{23}, f_{34}, f_{14}, r_{13}, r_{24})$ for some r_{24} in the interior of the desired interval. Let us call the arguments of the zero with smallest r_{24}, z_{ij}, so that $z_{34} = f_{34}(z_{24}, r_0), z_{14} = f_{14}(z_{24}, r_0)$, and $z_{24} = r_{24}$.

Note that the z_{ij} values are all positive and real by construction. If we solve the equation

$P(1, r_{23}, z_{34}, z_{14}, r_{13}, z_{24}) = 0$ for z_{34} we find that

$$z_{34}^2 = \frac{1}{2}\Big(r_{23}^2 + r_{13}^2 - 1 + z_{24}^2 + z_{14}^2 + (r_{13}^2 - r_{23}^2)(z_{24}^2 - z_{14}^2) \tag{7.6}$$
$$\pm 4D_4\sqrt{(1 + z_{24} + z_{14})(1 - z_{24} + z_{14})(1 + z_{24} - z_{14})(-1 + z_{24} + z_{14})}\Big).$$

The right hand side of this equation must also be real and positive. Within the square root, our assumptions guarantee that every term in parentheses is positive except the last term, $(-1 + z_{24} + z_{14})$. So this term must be non-negative. Also note that since $z_{34} < z_{24}$ by construction, the smaller root in (7.6) is the correct one, since the sum of the first four terms are already greater than z_{24}^2.

To see that the z_{ij} actually arise from a configuration of points in the plane, we consider a construction of such a configuration. The outer triangle is given. The circles of radius z_{14} and z_{24} around points 1 and 2, respectively, intersect in one or two points since we know that $z_{24} + z_{14} \geq 1$. Since the possible r_{34} must equal one of the values in (7.6), z_{34} must be one of these possible r_{34}. Figure 7.1 shows a typical case; the length of the dashed lines in the figure would be the two possible r_{34} values, of which z_{34} is the smaller as noted above.

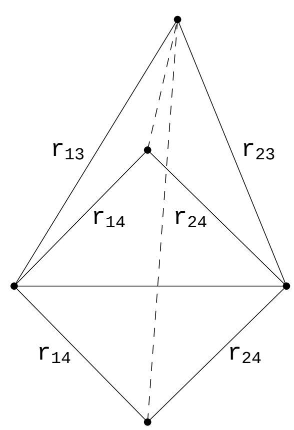


Figure 7.1: Example of possible r_{34} values

This concludes the proof of Theorem 7. □

The rest of this chapter consists of ingredients for the following theorem.

Theorem 8. *For a strictly admissible triangle with* $r_{23}^2 \geq r_{13}^2 - r_{13} + 1$ *and for* $r_0 \in (r_C, 1)$, *there is a unique concave central configuration for which the triangle is the outer triangle with parameter* r_0. *As the parameter* r_0 *is varied from* r_C *to* 1, *the fourth point traces out a simple curve from the circumcenter to the equilateral point* ($r_{14} = 1$, $r_{24} = 1$).

For a strictly admissible triangle with $r_{23}^2 < r_{13}^2 - r_{13} + 1$ *there exists an* $r_I \in (r_C, 1)$ *such that for each* $r_0 \in (r_C, r_I)$ *there is a unique concave central configuration for which the triangle is the outer triangle with parameter* r_0. *Furthermore, in the limit* $r_0 \to r_I$ *the fourth point of the configuration is collinear with the first and third points. As the parameter* r_0 *is varied from* r_C *to* r_I, *the fourth point traces out a simple curve.*

In both cases the position of the fourth point is a smooth function of r_0, r_{23}, *and* r_{13} *for* r_{23} *and* r_{13} *satisfying (7.1) and* $r_0 \in (r_C, 1)$ *or* $r_0 \in (r_C, r_I)$.

The proof of this theorem will be deferred until the end of the chapter, after all the necessary tools have been assembled.

Lemma 2. *For a given strictly admissible triangle and parameter* r_0 *such that* $r_C < r_0 < 1$, *there is at most one concave planar central configuration.*

Proof. Recall from the proof in Theorem 7 that we can find such central configurations by varying r_{24} in the interval (r_C, r_0) and finding zeros of $P(1, r_{23}, f_{34}, f_{14}, r_{13}, r_{24})$. Since f_{14} and f_{34} are monotonically increasing as functions of r_{24}, if there were two solutions corresponding to concave configurations one of them would have larger values of r_{24}, r_{14}, and r_{34} than the other. This in turn would imply that the values of D_1, D_2, and D_3 would all be larger in one of the solutions, since $\partial D_i / \partial r_{j4} = (r_{k4}^2 + r_{jk}^2 - r_{j4}^2)/(8 D_i r_{j4})$, $(i \neq j \neq k)$, which is greater than zero for a concave configuration in which $r_{j4} < r_{jk}$. But since the outer triangle is fixed, we must have $D_1 + D_2 + D_3 = D_4$ be constant, which is a contradiction, so there could not be two concave central configurations with the same outer triangle and value of r_0.

$\square$

The following simple lemma will be used several times:

Lemma 3. *For an admissible triangle with $r_{23} \neq 1$,*

$$\lim_{r_0 \to 1} z_{24} = \lim_{r_0 \to 1} z_{14} = 1.$$

Proof. This can be seen in many ways, but perhaps the simplest is to examine the original consistency equations (4.2). With our conventions it is immediate that $\lim_{r_0 \to 1} S_{12} = 0$. Since z_{34} is bounded away from 0 for a fixed third point in the interior of Q, all the terms in (4.2) must have zero as their limit. Furthermore, S_{13} and S_{23} are bounded away from zero (since $r_{23} > 1$), so $\lim_{r_0 \to 1} S_{24} = \lim_{r_0 \to 1} S_{14} = 0$, which is equivalent to the conclusion of the lemma. □

There is a corresponding lemma for the other endpoint of r_0 values:

Lemma 4. *For an admissible triangle*

$$\lim_{r_0 \to r_C} z_{24} = \lim_{r_0 \to r_C} z_{14} = r_C.$$

Proof. From the construction in the proof of Theorem 7 it is clear that $\lim_{r_0 \to r_C} z_{24} = r_C$. Examining the equations (4.2) we see that $\lim_{r_0 \to r_C} S_{34} = \lim_{r_0 \to r_C} S_{14} = 0$ since $\lim_{r_0 \to r_C} S_{24} = 0$ and the terms S_{12} and S_{23} are bounded away from zero for a fixed outer triangle. □

The configurations constructed in Theorem 6 are central configurations if their masses are positive. As described previously, a concave configuration which satisfies the consistency equations and our conventions will have positive masses if and only if the inequalities (6.1) are satisfied. These results simply follow from careful examination of (4.3). The last three relations of (6.1) hold by assumption, and the first three follow immediately by the previous construction. However, we must ensure that the configurations obtained are concave. In fact, the configurations will not always be concave if $r_{23}^2 < r_{13}^2 - r_{13} + 1$; the lemmas below will elucidate this point in more detail.

We also need to know something about the regularity of the functions z_{ij}.

Lemma 5. *For a planar solution to the consistency equations satisfying the conditions in Theorem 7 which is sufficiently close to concave,*

$$\frac{\partial}{\partial r_{24}}(P(1, r_{23}, f_{34}, f_{14}, r_{13}, r_{24})) > 0,$$

and so $z_{24}(r_{13}, r_{23}, r_0)$ is a smooth function.

Proof. For $i = 1, 2, 3$ we have

$$\frac{\partial P}{\partial r_{i4}}|_{P=0} = -64 r_{i4} \Delta_i \Delta_4 \tag{7.7}$$

from the investigations of Dziobek [D]. If the solution corresponds to a concave configuration, $\Delta_i < 0$, and we can compute

$$\frac{\partial}{\partial r_{24}}(P(1, r_{23}, f_{34}, f_{14}, r_{13}, r_{24})) =$$
$$\frac{\partial P}{\partial r_{24}}(1, r_{23}, f_{34}, f_{14}, r_{13}, r_{24}) + \frac{\partial P}{\partial r_{34}}\frac{\partial f_{34}}{\partial r_{24}} + \frac{\partial P}{\partial r_{14}}\frac{\partial f_{14}}{\partial r_{24}} =$$
$$64 D_4 (r_{24} D_2 + r_{34} D_3 \frac{S_{13} r_{34}^4}{S_{12} r_{24}^4} + r_{14} D_1 \frac{S_{13} r_{14}^4}{S_{23} r_{24}^4}) > 0.$$

Otherwise, the only Δ_i which could be positive is Δ_2, and it is zero for a configuration on the border of concavity. This means that if the configuration is sufficiently close to concave, the terms in the above expression will dominate and $\frac{\partial P}{\partial r_{i4}}|_{P=0} > 0$. $\square$

When the distances of the third point satisfy $r_{23}^2 < r_{13}^2 - r_{13} + 1$, which corresponds to the third point being below the diagonal in Figure 6.2, the configuration must cease to be concave before $r_0 = 1$. This follows from Lemma 3; the fourth point lies outside the convex hull of the first three points for r_0 sufficiently close to 1. But when $r_{23}^2 \geq r_{13}^2 - r_{13} + 1$ we must examine the curve of solutions in more detail.

Lemma 6. *For admissible outer triangles with $r_{12} = 1$, $r_{13} > 1$, and $r_{23}^2 \geq r_{13}^2 - r_{13} + 1$, the planar solutions of the consistency equations sufficiently close to $R_0 = R_{14} = R_{24} = 1$ with $R_0 < 1$ have $\theta_{214} < 60°$.*

Proof. The distances r_{23} and r_{13} of the outer triangle can be considered as parameters for the following argument (r_{12} is always equal to 1). The independent variables are R_0, R_{24}, and R_{14}. The last distance r_{34} could then be determined by choosing the root of the planarity condition $P = 0$ with the smallest value of r_{34}. It is easy to check that r_{34} is a C^1 function on the relevant domains. The consistency equations can be written as

$$g_1(R_0, R_{14}, R_{24}) = 0, \quad g_2(R_0, R_{14}, R_{24}) = 0 \tag{7.8}$$

where

$$g_1(R_0, R_{14}, R_{24}) = S_{23}S_{14} - S_{12}S_{34} \tag{7.9}$$

and

$$g_2(R_0, R_{14}, R_{24}) = S_{23}S_{14} - S_{13}S_{24}. \tag{7.10}$$

The Jacobian of the functions g_1 and g_2 is easily calculated:

$$D(g_1, g_2) =$$
$$\begin{bmatrix} (1 - R_{23} + R_{34} - R_{14}) & -(R_0 - R_{23} - (R_0 - 1)\frac{\partial R_{34}}{\partial R_{14}}) & (R_0 - 1)\frac{\partial R_{34}}{\partial R_{24}} \\ -(R_{23} - R_{13} + R_{14} - R_{24}) & -(R_0 - R_{23}) & (R_0 - R_{13}) \end{bmatrix}.$$

At the solution $R_0 = R_{14} = R_{24} = 1$ this reduces to

$$D(g_1, g_2)|_{(R_0=R_{14}=R_{24}=1)} = \begin{bmatrix} (R_{34} - R_{23}) & (R_{23} - 1) & (0) \\ (R_{13} - R_{23}) & (R_{23} - 1) & (1 - R_{13}) \end{bmatrix}. \tag{7.11}$$

Let us denote this Jacobian by $J_{(1,1,1)}$.

It is easy to see that $D(g_1, g_2)$ has rank 2 for any relevant values of the arguments; for instance, the fact that $R_{34} > R_{23}$ and $1 > R_{13}$ is sufficient. Thus there is a smooth curve $\mathcal{L}$ of solutions near $(1, 1, 1)$. The tangent of $\mathcal{L}$ at $(1, 1, 1)$ is simply the kernel of $J_{(1,1,1)}$. A little linear algebra yields

$$\begin{aligned} &\ker(J_{(1,1,1)}) = \\ &\operatorname{span}((1 - R_{23})(1 - R_{13}), (R_{34} - R_{23})(1 - R_{13}), (1 - R_{23})(R_{34} - R_{13})). \end{aligned} \tag{7.12}$$

Since none of the components of a kernel vector are zero, any of the quantities R_0, R_{14}, or R_{24} could be chosen as a parameter for $\mathcal{L}$. We will use R_{14} as the parameter, and let $Z_{24}(R_{14})$ denote the value of R_{24} along the curve $\mathcal{L}$. The explicit dependence on R_{14} will often be dropped. In order for nearby solutions to have $\theta_{214} < 60°$, they must satisfy the inequality $z_{24}^2 < z_{14}^2 - z_{14} + 1$. That inequality is equivalent to $dZ_{24}/dR_{14} > 1/2$ for solutions sufficiently close to $R_0 = R_{14} = R_{24} = 1$.

It is easy to compute from the above information that

$$\frac{dZ_{24}}{dR_{14}} = (\frac{1-R_{23}}{1-R_{13}})(\frac{R_{34}-R_{13}}{R_{34}-R_{23}}) \tag{7.13}$$

at the solution $R_0 = R_{14} = R_{24} = 1$.

Note that

$$(\frac{Z_{34}-R_{13}}{Z_{34}-R_{23}}) \geq 1, \tag{7.14}$$

where Z_{34} is the value of R_{34} on $\mathcal{L}$ determined by R_{14} and Z_{24}. The inequality (7.14) follows from $r_{23} \leq r_{13}$, so

$$\frac{dZ_{24}}{dR_{14}}|_{(R_0=R_{14}=Z_{24}=1)} \geq (\frac{1-R_{23}}{1-R_{13}}). \tag{7.15}$$

It will be shown that $(\frac{1-R_{23}}{1-R_{13}}) > 1/2$. First the case in which $r_{23}^2 = r_{13}^2 - r_{13} + 1$ will be proved. We consider r_{13} as a parameter. It is easy to see that $\lim_{r_{13}\to 1}(\frac{1-R_{23}}{1-R_{13}}) = \frac{1}{2}$. In fact that is the infimum of $(\frac{1-R_{23}}{1-R_{13}})$ for r_{13} in the interval $(1,2)$. To see this first observe that the inequality $(\frac{1-R_{23}}{1-R_{13}}) > \frac{1}{2}$ is equivalent to $0 < 1 + R_{13} - 2R_{23}$. Since this equality will be used in later chapters as well we will isolate it as a lemma:

Lemma 7. *For points in Q with $r_{23}^2 = r_{13}^2 - r_{13} + 1$, $1 + R_{13} - 2R_{23} > 0$.*

Proof. Substituting the expression for r_{23} into the right hand side of the inequality and taking the derivative with respect to the parameter r_{13} we obtain

$$\begin{aligned} & 3(-r_{13}^{-4} + (2r_{13}-1)(r_{13}^2 - r_{13} + 1)^{-5/2}) \\ > \ & 3(-r_{13}^{-4} + (2r_{13}-1)r_{13}^{-5}) \\ = \ & 3r_{13}^{-5}(r_{13}-1) \\ > \ & 0. \end{aligned} \tag{7.16}$$

The first inequality follows from the assumption that $r_{23} < r_{13}$ and the last inequality from the assumption that $r_{13} > 1$. Since the function $2R_{23}(r_{13}) - R_{13}$ is decreasing in the range we are considering, and equal to 1 in the limit $r_{13} \to 1$, it satisfies the desired inequality. □

To obtain the result for $r_{23}^2 > r_{13}^2 - r_{13} + 1$, we can simply note that $(\frac{1-R_{23}}{1-R_{13}})$ is a monotonically increasing function of r_{23} and a decreasing function of r_{13}.

This completes the proof of the lemma. □

Now it will be shown that the configurations obtained in the proof of Theorem 7 are always concave if $r_{23} \geq r_{13}^2 - r_{13} + 1$. Lemma 6 ensures that the curve of solutions parameterized by R_0 starts from $R_0 = R_{14} = R_{24} = 1$ with the angle of its tangent line greater than 60 degrees relative to the segment $\overline{12}$. The next lemma shows that the angle of the tangent line never becomes exactly 60 degrees, which means that $\theta_{214} \leq 60°$. This implies that the configurations stay concave since $\theta_{213} \geq 60°$ for configurations with $r_{23} \geq r_{13}^2 - r_{13} + 1$.

Lemma 8. *For a strictly admissible triangle in which the third point is above the diagonal, i.e.* $r_{23}^2 \geq r_{13}^2 - r_{13} + 1$*, there is a curve defined by the fourth point of the planar solutions of the consistency equations as the parameter* R_0 *is varied. The angle of the tangent line relative to the segment* $\overline{12}$ *is never equal to* 60 *degrees at any concave solution.*

Proof. The condition on the angle of the tangent is equivalent to

$$\frac{dr_{14}/dr_0}{dr_{24}/dr_0} \neq \frac{r_{24}(\sqrt{3(r_{24}^2 - (1 + r_{24}^2 - r_{14}^2)^2/4)} + \frac{1}{2}(1 - r_{24}^2 + r_{14}^2))}{r_{14}(\sqrt{3(r_{24}^2 - (1 + r_{24}^2 - r_{14}^2)^2/4)} - \frac{1}{2}(1 + r_{24}^2 - r_{14}^2))}. \tag{7.17}$$

This inequality is obtained by briefly reintroducing coordinates and substituting the interparticle distances into the condition on the angle of the tangent in these coordinates. We define coordinates by setting the coordinates of the first point equal to (-.5,0), the coordinates of the second point to (.5, 0), and requiring that the y-coordinates of the third and fourth points are positive. Then $r_{14}^2 = (x + .5)^2 + y^2$ and $r_{24}^2 = (x - .5)^2 + y^2$, where (x, y) are the coordinates of the fourth point. The desired condition on the tangent angle is equivalent to $\frac{dx}{dy} \neq \frac{1}{\sqrt{3}}$. Converting this back into the r_{ij} variables gives the condition

$$\frac{dx}{dy} = \frac{(\frac{dr_{14}}{dr_{24}} r_{14} - r_{24})(\sqrt{r_{24}^2 - (1 + r_{24}^2 - r_{14}^2)^2/4})}{r_{24} - (1 + r_{24}^2 - r_{14}^2)(2r_{24} - 2\frac{dr_{14}}{dr_{24}} r_{14})/4} \neq \frac{1}{\sqrt{3}}. \tag{7.18}$$

This is equivalent to (7.17) when the denominator is not zero. If the denominator is zero, that is still okay since then $\frac{dy}{dx} = 0 \neq \sqrt{3}$. The denominator and numerator cannot be zero simultaneously. If the numerator is equal to zero, then $\frac{dr_{14}}{dr_{24}} = r_{24}/r_{14}$ since the second factor in the numerator is actually equal to y, which is always positive. But in that case, the denominator would equal r_{24}, which cannot be zero.

We will show that the right hand side of (7.17) has a range contained in $(-\infty, -1) \cup (2, \infty)$ while the left hand side is always in $(-1, 2)$.

If the complicated expression on the right hand side of (7.17) is positive then it is always greater than $\frac{2r_{24}}{r_{14}}$ for $1/2 < r_{14} \leq r_{24} < 1$. To prove that, note that the denominator must be positive so we can cross-multiply to obtain the equivalent inequality

$$(r_{14}^2 - r_{24}^2) + (2\sqrt{3}y - 3) < 0, \tag{7.19}$$

since $y < \sqrt{3}/2$ the last term in parentheses in (7.19) is negative and the first term is non-positive. Here we have again used the fact that $y = \sqrt{r_{24}^2 - (1 + r_{24}^2 - r_{14}^2)^2/4}$.

If the right hand side of (7.17) is negative, then it will be less than $-r_{24}/r_{14}$ (which in turn is less than or equal to -1) if

$$r_{14}^2 - r_{24}^2 + 2\sqrt{3}y > 0. \tag{7.20}$$

Let us denote the left hand side of (7.20) by $h(r_{14}, r_{24})$. Since $\partial h/\partial r_{14} = r_{14}(2+\sqrt{3}(1+r_{24}^2 - r_{14}^2)/y)$ is positive for the relevant values of r_{14} and r_{24} we can check the boundary of the set of possible r_{14} and r_{24} to find the minimum of h. The lines $r_{14} = r_{24}$, $r_{24}^2 = r_{14}^2 - r_{14} + 1$, and $r_{14} = 1 + r_{24}^2 - \sqrt{3}r_{24}$ suffice for this computation; note that the last two correspond to $\theta_{214} = 60$ degrees and $\theta_{124} = 30$ degrees respectively. It is easy to compute that $h > 0$ on all of those lines; in fact the minimum of h is 1.

We need to show that $\frac{dr_{14}}{dr_{24}} < 2$, or $\frac{dR_{14}}{dR_{24}} < 2(\frac{r_{24}}{r_{14}})^4$. Throwing out the factor $(\frac{r_{24}}{r_{14}})^4$, the stronger inequality $\frac{dR_{14}}{dR_{24}} < 2$ will be proved. We will also see that $\frac{dR_{14}}{dR_{24}} > -1$ shortly.

To be in the kernel of $D(g_1, g_2)$ the ratio of the differentials dR_{14} and dR_{24} must satisfy:

$$\begin{aligned} &\frac{dR_{14}/dR_0}{dR_{24}/dR_0} = \\ &\frac{((1 - R_{23} + R_{34} - R_{14})(R_0 - R_{13}) + \frac{\partial R_{34}}{\partial R_{24}}(R_{23} - R_{13} + R_{14} - R_{24})(R_0 - 1))}{((1 + R_{34} - R_{13} - R_{24})(R_0 - R_{23}) - \frac{\partial R_{34}}{\partial R_{14}}(R_{23} - R_{13} + R_{14} - R_{24})(R_0 - 1))}. \end{aligned} \tag{7.21}$$

The partial derivatives of R_{34} will be obtained from the planarity condition. It will be shown below that $\frac{\partial R_{34}}{\partial R_{14}} < 0$ and $\frac{\partial R_{34}}{\partial R_{24}} < 0$. This makes it easy to see that $\frac{dR_{14}}{dR_{24}} > -1$ since if the right hand side of (7.21) is negative, it must be of the form $\frac{A-B}{C+B}$ with $A, B, C > 0$ and $B > A$.

The denominator of the right hand side of (7.21) is always positive in our domain so we can cross-multiply the desired inequality $\frac{dR_{14}}{dR_{24}} < 2$ to get the equivalent condition:

$$0 < (R_{23} + R_{14} - R_{13} - R_{24})(R_0 - 1)(-2\frac{\partial R_{34}}{\partial R_{14}} - \frac{\partial R_{34}}{\partial R_{24}})+ \\ 2(R_0 - R_{23})(1 + R_{34} - R_{13} - R_{24}) - (R_0 - R_{13})(1 + R_{34} - R_{23} - R_{14}) \tag{7.22}$$

To prove (7.22) we will show the first line of the left hand side is non-negative and the second line is positive. In the first term, everything is positive by assumption except the partial derivatives. Since $\frac{dR_{34}}{dR_{24}} = \frac{dr_{34}}{dr_{24}}(\frac{r_{24}}{r_{34}})^4$ and $\frac{dR_{34}}{dR_{14}} = \frac{dr_{34}}{dr_{14}}(\frac{r_{14}}{r_{34}})^4$, we can compute the sign of these derivatives for concave configurations by simplifying the expression obtained from solving $P = 0$ for r_{34} and then taking a derivative:

$$\begin{aligned} \frac{\partial r_{34}}{\partial r_{24}} &= \frac{r_{24}}{2r_{34}}(1 + r_{13}^2 - r_{23}^2 - D_4/D_3(1 + r_{14}^2 - r_{24}^2)) \\ &= \frac{r_{24}}{r_{34}}(r_{13}\cos\theta_{213} - D_4/D_3 r_{14}\cos\theta_{214}) \\ &= \frac{r_{24}r_{13}}{r_{34}\sin\theta 214}(\sin\theta_{214}\cos\theta_{213} - \sin\theta_{213}\cos\theta_{214}) \\ &= -\frac{r_{24}r_{13}}{r_{34}\sin\theta 214}(\sin\theta_{213} - \theta_{214}) \\ &= -D_2/D_3. \end{aligned} \tag{7.23}$$

To obtain the above result the law of cosines, Heron's formula, and the formulae $D_4 = r_{13}\sin\theta_{213}/2$ and $D_3 = r_{14}\sin\theta_{214}/2$ are used along with some trigonometric identities.

A very similar calculation yields

$$\frac{\partial r_{34}}{\partial r_{14}} = -D_1/D_3. \tag{7.24}$$

The second line in (7.22) can be rearranged as

$$\begin{aligned} & 2(R_0 - R_{23})(1 + R_{34} - R_{13} - R_{24}) - (R_0 - R_{13})(1 + R_{34} - R_{23} - R_{14}) \\ = \; & (R_0 - R_{23})(1 + R_{34} - 2R_{13} - 2R_{24} + R_{23} + R_{14}) - \\ & \qquad (R_{23} - R_{13})(1 + R_{34} - R_{23} - R_{14}) \\ \geq \; & (R_0 - R_{23})(1 + R_{34} - 2R_{13} + R_{23} - R_{14}) - \\ & \qquad (R_{23} - R_{13})(1 + R_{34} - R_{23} - R_{14}) \\ = \; & (R_0 - R_{23})(1 - R_{13}) + (R_{23} - R_{13})(R_0 - 1) + \\ & \qquad (R_{34} - R_{14})(R_0 + R_{13} - 2R_{23}). \end{aligned} \tag{7.25}$$

The inequality follows from $-2R_{24} + R_{14} \geq -R_{14}$. Only the last term in parentheses could be negative. However, simply note that $R_0 + R_{13} - 2R_{23} > 1 + R_{13} - 2R_{23}$. From Lemma 7 we already know that the latter function is positive on the diagonal, and since $1 + R_{13} - 2R_{23}$ is monotonically increasing as a function of r_{23} it is positive everywhere in Q above the diagonal as well, which completes the proof of Lemma 8. $\square$

Proof of Theorem 8. This theorem is basically a consequence of the earlier Lemmas in this chapter.

For the outer triangles with $r_{23}^2 \geq r_{13}^2 - r_{13} + 1$, we know from the analysis near the equilateral point with $r_{14} = 1$ and $r_{24} = 1$ that the fourth points of the solutions form a smooth curve $\mathcal{L}$ whose tangent angle is always greater than 60 degrees relative to the segment $\overline{12}$. Lemma 8 ensures that this tangent angle remains greater than 60 degrees. This in turn implies that the solution curve cannot become non-concave by leaving the region V on the line $\overline{13}$ or the boundary $r_{24} = 1$ since $\theta_{214} < 60°$(c.f. Figure 6.2). Since $\mathcal{L}$ is both closed and open in V it must end somewhere on the boundary of V. Note that it is impossible for $z_{24} = z_{14}$ for $r_{13} \neq r_{23}$ except for $z_{24} = z_{14} = z_{34} = r_0 = r_C$, by inspection of the consistency equations. Likewise the curve cannot hit the boundary where $r_{14} = r_{34}$ except at the circumcenter. So the curve must end at the circumcenter.

The situation is a little different when $r_{23}^2 < r_{13}^2 - r_{13} + 1$. There are concave solutions near the circumcenter where $r_{14} = r_C$ and $r_{24} = r_C$, and we know that the solutions obtained in Theorem 7 leave the region V at some parameter value $r_0 = r_I$. It is shown in Chapter 8 that this parameter value is continuous as a function of r_{23} and r_{13}. That proof also guarantees that the curve of solutions cannot re-enter the region V. Note that again it is impossible for the curve to hit the lines defined by $r_{24} = r_{14}$ or $r_{14} = r_{34}$ except at the circumcenter, so at $r_0 = r_I$ the fourth point is collinear with points 1 and 3.

In either case it is easy to see that $\mathcal{L}$ is a simple curve since at a self intersection point there would be two values of r_0 for the same configuration, which is impossible away from the equilateral point. $\square$

Chapter 8

BORDER CONFIGURATIONS

In this chapter the continuity of R_I is proved on the relevant domains. Recall that this is the value of the parameter R_0 at which points 1, 4, and 3 are collinear.

Definition 3. Let $w_{24}(r_{23}, r_{13}, r_{14}) = \sqrt{1 + r_{14}^2 - 2r_{14}(\frac{1+r_{13}^2-r_{23}^2}{2r_{13}})}$.

Definition 4. Let $w_{34}(r_{13}, r_{14}) = r_{13} - r_{14}$.

As before we will use the notation $W_{24} = w_{24}^{-3}$ and $W_{34} = w_{34}^{-3}$. Note that $(\frac{1+r_{13}^2-r_{23}^2}{2r_{13}}) = \cos\theta_{213}$.

Definition 5. Let $F_1 = \frac{W_{34}-R_{13}W_{24}}{1+W_{34}-R_{13}-W_{24}}$ and $F_2 = \frac{W_{34}-R_{23}R_{14}}{1+W_{34}-R_{23}-R_{14}}$.

Definition 6. Let $F(r_{23}, r_{13}, r_{14}) = F_1 - F_2$.

Note that $R_0 = F_1 = F_2$ is the condition for the consistency equations to be satisfied for the fourth point on the boundary of the exterior triangle.

Definition 7. Let Ω be the set of $(r_{23}, r_{13}, r_{14}) \in \mathbb{R}_+^3$ for which $1 < r_{23}^2 < r_{13}^2 - r_{13} + 1$, $r_{13}^2 < 1 + r_{23}^2$, and $r_{13}/2 < r_{14} < \frac{1}{2\cos\theta_{213}}$.

More explicitly, the variables in Ω must satisfy the conditions $\frac{1}{2} < \frac{r_{13}}{2} < r_{14} < \frac{1}{2\cos\theta_{213}} \leq 1 < r_{23} < r_{13} < 2$ as well as $(r_{13}^2 - 1) < r_{23}^2 < r_{13}^2 - r_{13} + 1$. The condition on r_{23} corresponds to the third point being below the diagonal in the region Q (cf. Figure 6.2).

Note that the denominators of both F_1 and F_2 are always positive on the domain Ω.

Definition 8. Configurations in which three points are collinear will be referred to as *border configurations*.

Definition 9. Let $\Pi(\Omega)$ be the projection of Ω onto its first two coordinates. I.e. $\Pi(\Omega)$ is the set of (r_{23}, r_{13}) satisfying the inequalities $1 < r_{23} < r_{13} < 2$ and $(r_{13}^2 - 1) < r_{23}^2 < r_{13}^2 - r_{13} + 1$.

The inequalities in the above definitions are somewhat redundant, in order to supply the reader with all the relevant relationships.

Theorem 9. *The set $F = 0$ in the domain Ω can be written as the graph of a continuous function $Y_{14}(r_{23}, r_{13})$ over $\Pi(\Omega)$.*

Proof. First we will show that there is at least one zero of F in Ω for any fixed r_{13} and r_{23}. We simply compute the values at the endpoints, and find that

$$F(r_{23}, r_{13}, \frac{r_{13}}{2}) = \frac{8R_{13} - R_{13}W_{24}}{1 + 7R_{13} - W_{24}} - 8R_{13}$$

$$= \frac{R_{13}(8 - W_{24} - 8(1 + 7R_{13} - W_{24}))}{1 + 7R_{13} - W_{24}}$$

$$= -\frac{7R_{13}(8R_{13} - W_{24})}{1 + 7R_{13} - W_{24}} < 0$$

and

$$F(r_{23}, r_{13}, \frac{1}{2\cos\theta_{213}}) = \frac{W_{34} - R_{13}R_{14}}{1 + W_{34} - R_{13} - R_{14}} - \frac{W_{34} - R_{23}R_{14}}{1 + W_{34} - R_{23} - R_{14}}$$

$$= \frac{(R_{23} - R_{13})(R_{14} - 1)(W_{34} - R_{14})}{(1 + W_{34} - R_{13} - R_{14})(1 + W_{34} - R_{23} - R_{14})} > 0$$

on Ω.

Next we compute the derivative of F with respect to r_{14}, at $F = 0$, in order to use the implicit function theorem:

$$
\begin{aligned}
\frac{dF}{dr_{14}}|_{F=0} = \;& 3(1+W_{34}-R_{13}-W_{24})^{-1}(1+W_{34}-R_{23}-R_{14})^{-1} \\
& [\;\, w_{34}^{-4}(R_0-1)(R_{23}+R_{14}-R_{13}-W_{24}) \\
& + r_{14}^{-4}(R_0-R_{23})(1+W_{34}-R_{13}-W_{24}) \\
& - w_{24}^{-5}(r_{14}-\cos\theta_{213})(R_0-R_{13})(1+W_{34}-R_{23}-R_{14})\;] \\
= \;& 3(1+W_{34}-R_{13}-W_{24})^{-1}(1+W_{34}-R_{23}-R_{14})^{-1} \\
& [\;\, w_{34}^{-4}(1+W_{34}-R_{23}-R_{14})^{-1}(R_{14}-1)\cdot \\
& \qquad\qquad (1-R_{23})(R_{23}+R_{14}-R_{13}-W_{24}) \\
& + r_{14}^{-4}(R_{14}-1)^{-1}(W_{34}-R_{23})(1-R_{13})(W_{24}-1) \\
& - w_{24}^{-5}(W_{34}-W_{24})^{-1}(r_{14}-\cos\theta_{213})\cdot \\
& \qquad\qquad (W_{34}-R_{23})(1-R_{13})(W_{34}-R_{14})\;]\,.
\end{aligned}
\tag{8.1}
$$

In order to obtain the last line of (8.1) we have used substitutions derived from the equation $F_1 = F_2$. As noted above, R_0 is the common value of F_1 and F_2 on the set $F = 0$. For the term with the coefficient r_{14}^{-4}, the substitution $R_0 = F_1$ is used, and for the term with the coefficient w_{24}^{-5} the substitution $R_0 = F_2$ is used. Then the resulting identity can be verified by expanding and rearranging

$$(W_{34}-R_{13}W_{24})(1+W_{34}-R_{23}-R_{14})-(W_{34}-R_{23}R_{14})(1+W_{34}-R_{13}-W_{24})=0$$

which is obtained by cross-multiplying the equation $F_1 = F_2$ by its denominators.

We will now show that (8.1) is positive on the domain Ω we are interested in. Every term in parentheses has been written so that it is positive on this domain, which makes it clear that the first two terms within the brackets are positive and the third is negative. We will see that the sum of the second and third terms is positive. After factoring out the common factor of $-(W_{34}-R_{23})(1-R_{13})$ we are left with the expression

$$-\frac{(W_{24}-1)}{r_{14}^4(R_{14}-1)}+\frac{(r_{14}-\cos\theta_{213})(W_{34}-R_{14})}{w_{24}^5(W_{34}-W_{24})}. \tag{8.2}$$

After factoring out some positive terms and clearing some fractions, the sign of (8.2) is determined by the sign of

$$-r_{14}^2(1-w_{24}^3)(w_{24}^3-w_{34}^3)+w_{24}(r_{14}-\cos\theta_{213})(1-r_{14}^3)(r_{14}^3-r_{34}^3). \tag{8.3}$$

Now let us make a change of variables from (r_{14}, r_{13}, r_{23}) to (r_{14}, r_{13}, α), where $\alpha = \cos\theta_{213}$. Then w_{24} is a function of r_{14} and α, whereas w_{34} is function of r_{14} and r_{13}. Let us denote the function of these arguments in (8.3) by $G(r_{14}, r_{13}, \alpha)$.

In our new variables, the inequalities defining Ω can be written as $\frac{1}{2} < \alpha < \frac{1}{\sqrt{2}}$, $2\alpha < r_{13} < \frac{1}{\alpha}$, and $\frac{r_{13}}{2} < r_{14} < \frac{1}{2\alpha}$.

A simple computation yields that

$$\frac{\partial G}{\partial r_{13}} = 3(r_{13}-r_{14})^2(r_{14}^2+w_{24}(1+r_{14})(\alpha(1-r_{14}+r_{14}^2)-r_{14})) \tag{8.4}$$

which will always be positive if $G_1 = r_{14}^2+w_{24}(1+r_{14})(\alpha(1-r_{14}+r_{14}^2)-r_{14})$ is positive on Ω.

Since $\frac{\partial G_1}{\partial \alpha} = (1+r_{14}^2(2+2r_{14}+r_{14}^3)-3\alpha(r_{14}+r_{14}^4))/w_{24}$, the sign of $\frac{\partial G_1}{\partial \alpha}$ is determined by the sign of

$$G_2(r_{14},\alpha) = 1+r_{14}^2(2+2r_{14}+r_{14}^3)-3\alpha(r_{14}+r_{14}^4).$$

It is easy to see that $\frac{\partial G_2}{\partial \alpha} < 0$, so to compute a lower bound for $G_2(r_{14},\alpha)$ we can examine $G_2(r_{14}, 1/\sqrt{2})$.

Now if we look at $\frac{\partial G_2}{\partial r_{14}}(r_{14}, 1/\sqrt{2})$ we get a quartic in r_{14}:

$$q(r_{14}) = -\frac{3\sqrt{2}}{2}+4r_{14}+6r_{14}^2-6\sqrt{2}r_{14}^3+5r_{14}^4 \tag{8.5}$$

and the only real roots of this quartic occur at r_{14} less than $1/2$. There are many ways to see this, but one elementary way is to notice that the second derivative of q is $12(5r_{14}^2-3\sqrt{2}r_{14}+1)$ which is always positive, and since $\frac{dq}{dr_{14}}|_{r_{14}=\frac{1}{2}} = (25-9\sqrt{2})/2 > 0$ and $q(\frac{1}{2}) = \frac{61-36\sqrt{2}}{4} > 0$ the polynomial q is positive for all $r_{14} > \frac{1}{2}$.

Thus the sign of $\frac{\partial G_2}{\partial r_{14}}$ is positive on Ω. So the minimum of G_2 occurs at $r_{14} = r_{13}/2$ and $\alpha = 1/\sqrt{2}$. This lets us compute $G_2 \geq G_2(r_{13}/2, 1/\sqrt{2}) > G_2(1/2, 1/\sqrt{2}) = \frac{57-27\sqrt{2}}{32} > 0$. So $\frac{\partial G_1}{\partial \alpha} > 0$, which implies that $G_1(r_{14},\alpha) > G_1(r_{14}, 1/2)$.

We now need to show that $G_1(r_{14}, 1/2) = r_{14}^2 + (1+r_{14})\sqrt{1 - r_{14} + r_{14}^2}(1-3r_{14}+r_{14}^2)/2$ is always positive on the interval $(\frac{1}{2}, 1)$. The argument is by contradiction. If $G_1(r_{14}, 1/2) = 0$ then the following polynomial $p(r_{14})$ would also be zero at the same value of r_{14}

$$p_1(r_{14}) = (1 - r_{14})^2(1 - 3r_{14} - 2r_{14}^2 + 5r_{14}^3 - 2r_{14}^4 - 3r_{14}^5 + r_{14}^6).$$

This is obtained by rearranging, squaring, and simplifying the equation $G_1(r_{14}, 1/2) = 0$. Let $p(r_{14}) = 1 - 3r_{14} - 2r_{14}^2 + 5r_{14}^3 - 2r_{14}^4 - 3r_{14}^5 + r_{14}^6$. Now we compute that $\frac{d^2p}{dr_{14}^2} = -4 + 30r_{14} - 24r_{14}^2 - 60r_{14}^3 + 30r_{14}^4$. Computing a few values of the quartic $\frac{d^2p}{dr_{14}^2}$:

r_{14}	-1	0	$\frac{1}{3}$	$\frac{1}{2}$	2	3
$\frac{d^2p}{dr_{14}^2}\vert_{r_{14}}$	32	-4	$\frac{40}{27}$	$-\frac{5}{8}$	-40	680

shows that there are four real roots of the second derivative outside the interval $[\frac{1}{2}, 1]$ so $\frac{d^2p}{dr_{14}^2} < 0$ in that interval. Since $\frac{dp}{dr_{14}}|_{r_{14}=\frac{1}{2}} = -3$ we also have that $\frac{dp}{dr_{14}} < 0$ on the interval. Finally, since $p(\frac{1}{2}) = -37/64 < 0$, we see that p cannot be zero in the interval $[\frac{1}{2}, 1]$. So p_1 is also never zero in the interval, which is a contradiction.

So G_1 is positive on Ω, which implies that $\frac{\partial G}{\partial r_{13}}$ is positive on the domain Ω. So G will be less than its values on the boundary of Ω with largest r_{13}, which is when $r_{13} = 2r_{14}$. It is easy to see from (8.3) that $G \leq 0$ at those points since $w_{34} = r_{14}$, so it must be negative on the interior of the domain Ω. This in turn implies that $\frac{dF}{dr_{14}}|_{F=0} > 0$ so there is exactly one zero of F for each $(r_{23}, r_{13}) \in \Pi(\Omega)$. The implicit function theorem then lets us describe the set $F = 0$ as the continuous graph of $Y_{14}(r_{23}, r_{13})$ over $\Pi(\Omega)$.

□

Since $R_0 = F_1 = F_2$ on the set $F = 0$, Theorem 9 implies that there is a function $R_I(r_{23}, r_{13})$ which is continuous in $\Pi(\Omega)$, and such that for $R_0 = R_I(r_{23}, r_{13})$ there is a configuration satisfying the consistency equations with the fourth point collinear with first and third points.

We will also need to understand the behavior of the function R_I on the boundary of the region $\Pi(\Omega)$.

The easiest case is when $r_{13}^2 = r_{23}^2 + 1$. Here the required interval for r_{14} shrinks to $r_{13}/2 = r_C$, which means

$$\lim_{r_{13} \to \sqrt{r_{23}^2+1} = \frac{1}{\alpha}} Y_{14} = R_C. \tag{8.6}$$

This in turn means that we can simply extend the value of R_I to be R_C on these boundary points.

The next type of boundary point we will examine is a point $p_0 = (r_{23}, r_{13})$ such that $r_{23} = 1$ and $r_{13} \neq 1$. As one would hope and expect, as these isosceles configurations are approached the limit of Y_{14} equals its value in the isosceles case ($(r_{13}/2)^{-3} = 8R_{13}$), as the following lemma shows.

Lemma 9. $\lim_{p \to (1, r_{13})} Y_{14} = 8R_{13}$ *for* $r_{13} \neq 1$ *and* $p \in \Pi(\Omega)$.

Proof. Let us write the components of p as $(\tilde{r}_{23}, \tilde{r}_{13})$ in coordinates (r_{23}, r_{13}). It has already been noted that $F(\tilde{r}_{23}, \tilde{r}_{13}, \tilde{r}_{13}/2) < 0$ (in the beginning of the proof of Theorem 9). Now it will be shown that $F(1 + \epsilon^2, \tilde{r}_{13}, \tilde{r}_{13}/2 + \epsilon + O(\epsilon^2)) > 0$ for sufficiently small ϵ, which implies the lemma since for each fixed $\tilde{r}_{23}$ and $\tilde{r}_{13}$ there is only one zero of F in the interval $(\tilde{r}_{13}/2, \frac{1}{2\cos\theta_{213}})$.

Instead of using ϵ, let us work in the capital variables (the inverse cubes) with $\tilde{R}_{23} = 1 - E_{23}$ and $R_{14} = 8\tilde{R}_{13} - E_{14}$. Then we have $W_{34} = 8\tilde{R}_{13} + E_{14} + O(E_{14}^2)$ and F_2 becomes

$$\frac{2E_{14} + 8\tilde{R}_{13}E_{23} - E_{14}E_{23} + O(E_{14}^2)}{2E_{14} + E_{23} + O(E_{14}^2)}.$$

If we let $E_{14} = \sqrt{E_{23}}$, then $F_2 = 1 + O(E_{14})$ and since $F_1 > 1$ (even at $E_{14} = 0$) we have $F > 0$ for sufficiently small E_{14}. So for sufficiently small E_{14} there will be a zero of F for R_{14} in the interval $(8\tilde{R}_{13}, 8\tilde{R}_{13} + E_{14})$. If we choose $\epsilon = \tilde{r}_{13}^4 E_{14}/48$ we have $F(1 + \epsilon^2, \tilde{r}_{13}, \tilde{r}_{13}/2 + \epsilon + O(\epsilon^2)) > 0$ as desired.

□

The penultimate category of boundary point consists of p_0 for which $\alpha = 1/2$, or equivalently when $r_{23}^2 = r_{13}^2 - r_{13} + 1$. We have already examined this case from a somewhat different perspective, in Lemma 8. The calculations in that lemma suggest that $\lim_{p \to p_0} Y_{14} = 1$. The following lemma makes that intuition precise.

Lemma 10. *For $r_{13} \neq 1$, $\lim_{p \to p_0} Y_{14} = 1$, where $p_0 = (\sqrt{r_{13}^2 - r_{13} + 1}, r_{13})$.*

Proof. Recall that for each $p = (r_{23}, r_{13})$ we define Y_{14} as the unique value of $r_{14} \in (r_{13}/2, 1/2\alpha)$ for which $F(r_{23}, r_{13}, r_{14}) = 0$.

In order to examine where F changes sign, we expand around the point where $R_{14} = W_{24} = (2\alpha)^3$, writing $R_{14} = (2\alpha)^3 + E_{14} = 1 + E_\alpha + E_{14}$ and $W_{24} = (2\alpha)^3 + E_{24} = 1 + E_\alpha + E_{24}$. For a border configuration it is easy to show that in fact $1 + E_\alpha + E_{24} = 1 + E_\alpha + E_{14}/2 + O(E_{14}^2)$.

Now consider the sign of $F(r_{23}, r_{13}, 1 + E_\alpha + E_{14})$. After clearing the positive denominators, the sign of F is determined by

$$\begin{aligned} &E_\alpha(R_{23} - R_{13})(W_{34} - 1 - 3E_{14}/2 - E_\alpha) \\ &-E_{14}(W_{34}(R_{13} + 1 - 2R_{23}) + (R_{23} + R_{23}R_{13} - 2R_{13}))/2 \\ &+O(E_{14}^2). \end{aligned} \tag{8.7}$$

For p sufficiently close to the diagonal (where $\alpha = 1/2$) note that $R_{13} + 1 - 2R_{23} > 0$ by Lemma 7. Since $W_{34} \geq 1$, we have $W_{34}(R_{13} + 1 - 2R_{23}) + (R_{23} + R_{23}R_{13} - 2R_{13}) \geq (1 - R_{23})(1 - R_{13}) > 0$. For sufficiently small E_α, this means that $F(r_{23}, r_{13}, 1 + E_\alpha + E_{14})$ changes sign at

$$E_{14} = E_\alpha \frac{2(R_{23} - R_{13})((r_{13} - 1)^{-3} - 1)}{((r_{13} - 1)^{-3}(R_{13} + 1 - 2R_{23}) + (R_{23} + R_{23}R_{13} - 2R_{13}))} + O(\alpha^2). \tag{8.8}$$

The coefficient of E_α in (8.8) is nonzero for $r_{13} \neq 1$, so the zero of F is located at $1 + O(E_\alpha)$, which proves the lemma.

□

Finally, we will examine R_I on the blowup of the point where $(r_{23}, r_{13}) = (1, 1)$. For this we introduce $c = \frac{1 - R_{23}}{1 - R_{13}}$. For admissible outer triangles we have $c \in (0, 1/2)$, and (c, r_{13}) may be used as coordinates instead of (r_{23}, r_{13}).

Theorem 10. *For admissible outer triangles there is a continuous function $Y_{14b}(r_{13}, c)$ defining the border configurations near the blowup $r_{23} = r_{13} = 1$.*

Proof. The consistency equations for a border configuration on the blowup can be written, using the variables c and $E_{13} = 1 - R_{13}$, as

$$(W_{34} - R_{14})(W_{24} - 1) - c(W_{34} - W_{24})(R_{14} - 1) + cE_{13}(R_{14} - R_{24}) = 0. \tag{8.9}$$

Consider instead the above equation with $E_{13} = 0$.

$$(W_{34} - R_{14})(W_{24} - 1) - c(W_{34} - W_{24})(R_{14} - 1) = 0, \tag{8.10}$$

Let us denote the function in (8.10) as F_b. We will show that at the relevant zeros of F_b its derivative is non-zero. This property is stable under perturbations, and so the function in (8.9) will also have that property for sufficiently small E_{13}.

There is at least one zero of F_b with r_{14} in the interval $(1/2, 1)$, for $c \in (0, 1/2)$. To see this we simply compute values at the endpoints. At $r_{14} = 1/2$, we have $R_{14} = W_{34} = 8$ and $F_b = -56c(1 - 3^{-3}) < 0$. At the other end, W_{34} becomes infinite and we must take a limit. If $r_{14} = 1 - \epsilon_{14}$ for $0 < \epsilon_{14} << 1$, then $R_{14} \approx 1 + 3\epsilon_{14}$ and $W_{24} \approx 1 + 3\epsilon_{14}/2$. The coefficient of W_{34} will determine the sign of F_b in the limit as $\epsilon_{14} \to 0$, and it is $3\epsilon_{14}(1/2 - c) > 0$ to first order in ϵ_{14}.

Now we turn to the derivative of F_b. We can solve for c in the equations $F_b = 0$ and plug it into $\frac{dF_b}{dr_{14}}$ to obtain

$$\begin{aligned}\frac{dF_b}{dr_{14}}|_{F_b=0, r_{13}=1} = 3\,[\, & w_{24}^3(2 - r_{14})(1 + r_{14})(1 - r_{14}(1 - r_{14})(4 - 7r_{14} + 7r_{14}^2)) - \\ & 2(1 - r_{14}(1 - r_{14})(5 + r_{14}(1 - r_{14})(-8 + r_{14}^2(1 - r_{14})^2)(3 - r_{14} + r_{14}^2)))\,]\,/ \\ & (2((1 - r_{14})^4(1 + r_{14} + r_{14}^2) - w_{24}^3(1 - r_{14}^3))).\end{aligned} \tag{8.11}$$

We would like to show that the derivative $\frac{dF_b}{dr_{14}}|_{F_b=0}$ is not zero for $r_{14} \in (1/2, 1)$. First we show that the denominator in (8.11) is never zero on $(1/2, 1)$. Any zero of the denominator would also be a zero of the polynomial

$$r_{14}(1 + r_{14} + r_{14}^2)^2(3 - 9r_{14} + 13r_{14}^2 - 9r_{14}^3 + 3r_{14}^4) \tag{8.12}$$

which is obtained by setting the denominator to zero, rearranging and squaring to get rid of the square root in W_{24}, and factoring. It is easy to see that any zeros in $(1/2, 1)$ would have to come from the quartic factor p_d in (8.12). But the symmetry of the coefficients of

p_d implies that if there were a zero in $(1/2, 1)$, there would also be a zero in $(1, 2)$. This is impossible since $p_d(1) = 1$, $\frac{dp_d}{dr_{14}}(1) = 2$, and $\frac{d^2 p_d}{dr_{14}^2} > 0$ on $(1/2, 1)$.

Now it must be shown that the numerator of (8.11) has no zeros in the interval. We use the same technique of setting the numerator to zero, and rearranging and squaring to eliminate the square roots to obtain a polynomial in r_{14} which has all the zeros of the numerator. If we ignore factors which do not have zeros in the interval in question, and we make the change of variables $z = (r_{14} - 1/2)^2$, we are left with the polynomial

$$\begin{aligned} p_n(z) = 32768z^8 - 270336z^7 + 653312z^6 + 1041920z^5 - 1603968z^4 \\ -2648672z^3 - 892760z^2 - 184702z - 23569. \end{aligned} \tag{8.13}$$

On $(0, 1/4)$, $p_n(z)$ is less than the value obtained by evaluating all the positive terms in $p_n(z)$ at zero and all the negative terms at $z = 1/4$. Since this value is -22391.5, $p_n(z)$ has no zeros on $(0, 1/4)$. This is equivalent to the numerator of (8.11) having no zeros in $(1/2, 1)$.

The implicit function theorem then guarantees the continuity of the resulting function $Y_{14b}(c, r_{13})$. □

We will need Theorem 10 in Chapter 10, where its relevance will be explained.

Chapter 9

TOPOLOGICAL LEMMA

In many of the arguments in the following chapters, and indeed for the main result of this paper, a topological lemma will be used. This can be thought of as one possible extension of the intermediate value theorem to higher dimensions. Note that here, and throughout this paper, D^n is the closed unit disk in $\mathbb{R}^n$.

Lemma 11. *A continuous map $f : D^n \to \mathbb{R}^n$, that restricts to a degree one map $f|_{S^{n-1}} : S^{n-1} \to S^{n-1}$, is surjective onto D^n.*

Proof. The proof is by contradiction. Assume that the map is not surjective onto D^n. Then at least one point, p, is not in the image. However, the induced map on the homology groups is the identity on $H_{n-1}(S^{n-1}) = \mathbb{Z}$. As seen in the commutative diagram below, this is impossible since the map factors through 0.

$$\begin{tikzcd} S^{n-1} \arrow[r, "f|_{S^{n-1}}"'] \arrow[d, "i"'] & S^{n-1} \arrow[d, "i"'] \\ D^n \arrow[r, "f"] & \mathbb{R}^n - \{p\} \end{tikzcd}$$

$$\begin{tikzcd} H_{n-1}(S^{n-1}) = \mathbb{Z} \arrow[r, "f_*|_{S^{n-1}}=Id"'] \arrow[d, "i_*"'] & H_{n-1}(S^{n-1}) = \mathbb{Z} \arrow[d, "Id"'] \\ H_{n-1}(D^n) = 0 \arrow[r, "f_*"] & H_{n-1}(\mathbb{R}^n - \{p\}) = \mathbb{Z} \end{tikzcd}$$

∎

For more information on differential topology and the degree of a map see the excellent introduction by Hirsch [Hr].

Chapter 10

EXTENSION OF THE MASS MAP

The goal of this chapter is to apply Lemma 11 to the map from the set of configurations which are canonical concave central configurations to the space of masses.

Definition 10. Let the set T be defined as the $(r_{23}, r_{13}, r_0) \in \mathbb{R}^3_+$ for which r_{23} and r_{13} are determined by a point in Q (see (7.1)), $r_0 \in (r_C, r_I)$ for $r_{23}^2 \leq r_{13}^2 - r_{13} + 1$, and $r_0 \in (r_C, 1)$ for $r_{23}^2 \geq r_{13}^2 - r_{13} + 1$.

Recall that $\mathcal{M} = \{(m_1, m_2, m_3, m_4) | \sum m_i = 1,\, m_i \geq 0\}$.

Definition 11. Let $\mathcal{M}_c = \{(m_1, m_2, m_3, m_4) | \sum m_i = 1,\, m_i \geq 0, m_1 \geq m_2, m_1 \geq m_3\}$.

Definition 12. Let $\widetilde{\mathcal{M}_c}$ be the union of $\mathcal{M}_c$ and the five subsets of $\mathcal{M}$ given by $m_4 = 0$, $m_2 = 0$, $m_3 = 0$, $m_1 = m_2$, and $m_1 = m_3$.

Definition 13. Let $\sigma_{ji} = \frac{\Delta_j S_{ik}}{\Delta_i S_{jk}}$, for $1 \leq k \leq 4$, $k \neq i$, and $k \neq j$. The Z_{i4} variables are to be substituted for the R_{i4} variables in these expressions.

Note that the σ_{ji} are simply the mass ratios as given in (4.3).

Recall that $Z_{i4} = z_{i4}^{-3}$ are the smooth functions of r_0, r_{23}, and r_{13} constructed in Chapter 7 which give the distances from the points of the outer triangle to the (interior) fourth point. By construction configurations with these distances are concave central configurations.

Definition 14. The mass map $\mathcal{F} : T \to \mathcal{M}$ is defined by $m_i = 1/(1 + \sum_{j \neq i} \sigma_{ji})$.

The set T parameterizes the set of concave central configurations. We would like to extend the mass map to the boundary of this set, but this is not possible on the natural closure of T in $\mathbb{R}^3$. We must at least blow up $\overline{T}$ at the point $(1, 1, 1)$.

Definition 15. In coordinates (r, θ, ϕ), define $U = [0, \infty) \times [0, \pi/2] \times [0, \pi/4]$.

Definition 16. Let $\beta : U \to \mathbb{R}^3$ be the blow-down map given by $\beta(r, \theta, \phi) \mapsto (1 + r\tan(\phi)\cos(\theta), 1 + r\cos(\theta), 1 - r\sin(\theta))$.

Definition 17. Let $T_b = \overline{\beta^{-1}(T)}$.

In order to apply Lemma 11 we must first prove the following theorem:

Theorem 11. *The map $\mathcal{F}$ extends continuously to the boundary of T_b.*

Since the proof of Theorem 11 is rather long, it will be broken up into sections 10.1 - 10.9. The remainder of this section will lay the groundwork for these arguments.

Instead of (r, θ, ϕ), we will use local coordinates (r_{23}, r_{13}, r_0), (r_0, a, c), (r_{13}, d, c), or (r_0, a, b), where $a = (1-R_{13})/(R_0-1)$, $b = (1-R_{23})/(R_0-1)$, $c = (1-R_{23})/(1-R_{13})$, and $d = a^{-1} = (R_0-1)/(1-R_{13})$. Although a, b, c, and d are defined in terms of (r_{23}, r_{13}, r_0) they extend smoothly to most of T_b. In particular, a and b are smooth everywhere on T_b except where $\theta = 0$ (face 7), d smoothly extends everywhere but $\theta = \pi/2$, and c extends smoothly to all of T_b. Note that $a = \cot(\theta) + O(r)$, $b = \tan(\phi)\cot(\theta) + O(r)$, $c = \tan(\phi) + O(r)$, and $d = \tan(\theta) + O(r)$.

Definition 18. For $(r\cos(\theta)\cos(2\phi)) \geq \cos^2(\phi)(2\tan(\phi) - 1)$ let $d_i(r_{13}, c) = (W_{24} - R_I)/(W_{34} - W_{24})$.

Note that the condition $(r\cos(\theta)\cos(2\phi)) \geq \cos^2(\phi)(2\tan(\phi) - 1)$ is simply the inequality $r_{23}^2 \geq r_{13}^2 - r_{13} + 1$ extended to the blowup. Outer triangles satisfying the inequality have their third point above the diagonal in Q (cf. Figure 6.2). The arguments in Chapter 8 insure that Y_{14} is a continuous function of r_{13} and c. Note also that some functional dependence was suppressed in Definition 18; R_I and Y_{14} are functions of r_{13} and c, and W_{24} and W_{34} are functions of Y_{14} and r_{13}.

Lemma 12. *The set T_b is homeomorphic to D^3, and its boundary consists of the following seven two-dimensional pieces:*

1. *A face T_1 contained in the plane $\phi = \pi/4$ whose image points under β have $r_{23} = r_{13}$.*

2. *A face T_2 contained in the plane $\phi = 0$ whose image points under β have $r_{23} = 1$.*

3. *A face* T_3 *contained in the plane* $\theta = \pi/2$ *whose image points under* β *have* $r_{23} = r_{13} = 1$.

4. *A surface* T_4 *whose image points under* β *have* $r_0 = r_I$ *and* $r_{23}^2 < r_{13}^2 = r_{13}^2 - r_{13} + 1$.

5. *A face* T_5 *contained in the plane* $r = 0$.

6. *A surface* T_6 *whose image points under* β *have* $r_0 = r_C$.

7. *A face* T_7 *contained in the plane* $\theta = 0$ *whose image points under* β *have* $r_{23}^2 \geq r_{13}^2 = r_{13}^2 - r_{13} + 1$ *and* $r_0 = 1$.

Proof. The fact that T_b is homeomorphic to a closed ball follows immediately from the fact that β^{-1} is a diffeomorphism on the interior and T^{circ} is homeomorphic to a ball. Likewise, the pre-images of boundary faces T_1, T_2, T_4, T_6, and T_7 are all mapped diffeomorphically to the boundary of T_b except for their boundary components which have $r_{23} = r_{13} = 1$. We will use the word *face* for all the boundary components T_i even they are not all polygons.

It is not hard to compute the properties of the remaining faces, T_3 and T_5. For later use we will explicitly describe the all the edges T_{ij} and vertices T_{ijk} where two or three of the faces intersect.

There are ten vertices where three of the faces meet. The following list names the vertices by which faces they are boundaries of, and then gives their coordinates (r, θ, ϕ).

- Faces T_1, T_6, T_7 meet at $T_{167} = (\sqrt{2+\sqrt{3}} - 1, 0, \pi/4)$.

- Faces T_1, T_3, T_6 meet at $T_{136} = (1 - \frac{1}{\sqrt{3}}, \pi/2, \pi/4)$.

- Faces T_2, T_3, T_6 meet at $T_{236} = (1 - \frac{1}{\sqrt{3}}, \pi/2, 0)$.

- Faces T_2, T_4, T_6 meet at $T_{246} = (\sqrt{3}(1 - \frac{1}{\sqrt{2}}), \arccos(\sqrt{\frac{2}{3}}), 0)$.

- Faces T_4, T_6, T_7 meet at $T_{467} = (1, 0, \arctan(\sqrt{3} - 1))$.

- Faces T_1, T_5, T_7 meet at $T_{157} = (0, 0, \pi/4)$.

- Faces T_1, T_3, T_5 meet at $T_{135} = (0, \pi/2, \pi/4)$.
- Faces T_2, T_3, T_5 meet at $T_{235} = (0, \pi/2, 0)$.
- Faces T_2, T_4, T_5 meet at $T_{245} = (0, \arctan(\frac{21\sqrt{3}-19)}{208}, 0)$.
- Faces T_4, T_5, T_7 meet at $T_{457} = (0, 0, \arctan(\frac{1}{2}))$.

There are fifteen edges T_{ij}. Eight of these are simply line segments between their vertices, namely T_{23}, T_{13}, T_{36}, T_{35}, T_{25}, T_{57}, T_{15}, and T_{17}.

The edge T_{24} is given by $r_{23} = 1$ and $r_0 = r_I(1, r_{13})$, where

$$r_I(1, r_{13}) = \frac{8(r_{13}^6 - 12r_{13}^4 + 48r_{13}^2 - 56)}{r_{13}^9 - 12r_{13}^7 + 7r_{13}^6 + 48r_{13}^5 - 84r_{13}^4 + 336r_{13}^2 - 448}.$$

The edge T_{47} consists of points $(\sqrt{r_{13}^2 - r_{13} + 1}, r_{13}, 1)$ in coordinates (r_{23}, r_{13}, r_0).

The edge T_{16} is given by $r_{23} = r_{13}$ and $r_0 = r_C$. For these configurations $r_C = r_{13}^2/\sqrt{4r_{13}^2 - 1}$.

Similarly T_{26} consists of points $(1, r_{13}, r_C)$ in coordinates (r_{23}, r_{13}, r_0). In this case $r_C = 1/\sqrt{4 - r_{13}^2}$.

For the edge T_{46} we can also use (r_{23}, r_{13}, r_0) coordinates, with points $(\sqrt{r_{13}^2 - 1}, r_{13}, r_{13}/2)$. These configurations have right-angled outer triangles with $r_0 = r_I = r_C$.

The edge T_{67} consists of points $(\sqrt{3}r_{13}/2 + \sqrt{4 - r_{13}^2}/2)$ in coordinates (r_{23}, r_{13}, r_0).

The only edge for which we lack an explicit description is T_{45}. In coordinates (r_{13}, d, c), points on this edge have the form $(1, d_i(c), c)$.

$\square$

Our goal is to extend the mass map $\mathcal{F}$ to T_b. We will take this one face of the boundary of T_b at a time, starting with the isosceles boundaries. After examining the interior of each face we will consider each edge and then each vertex. Each face of the boundary is mapped by the extension to a point, a line, or a face of the boundary of $\widetilde{\mathcal{M}}_c \subset \mathcal{M}$. Note that for some convenience of exposition, T has been blown up more than is strictly necessary for the purpose of extending $\mathcal{F}$; i.e. T_b is slightly larger than necessary.

In most cases the first step will be to extend the functions $z_{i4}(r_{23}, r_{13}, r_0)$, the interior interparticle distances, to the boundary points in question. Then the extension of $\mathcal{F}$ will be examined.

We will commit a mild abuse of notation by denoting the extension of $\mathcal{F}$ by $\mathcal{F}$ as well.

A schematic diagram of T_b is shown in Figure 10.1. The vertices are accurate; two of the faces are depicted as flat although they are curved surfaces. These will be discussed in more detail below. Figure 10.2 is an accurate depiction of some of the curved boundary lines, and includes the image of the diagonal in Q where $r_{23}^2 = r_{13}^2 - r_{13} + 1$.

For reference we list here the most frequently used expressions for the components of $\mathcal{F}$.

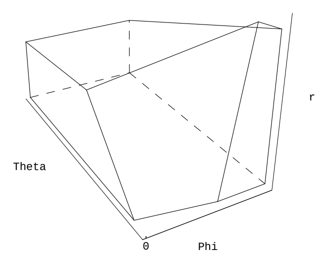


Figure 10.1: Schematic of T_b structure

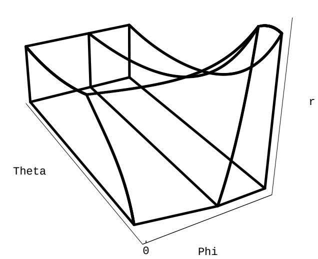


Figure 10.2: T_b structure showing diagonal

$$m_1 = \left(1 + \frac{\Delta_2 S_{13}}{\Delta_1 S_{23}} + \frac{\Delta_3 S_{12}}{\Delta_1 S_{23}} + \frac{\Delta_4 S_{13}}{\Delta_1 S_{34}}\right)^{-1}, \tag{10.1}$$

$$m_2 = \left(1 + \frac{\Delta_1 S_{23}}{\Delta_2 S_{13}} + \frac{\Delta_3 S_{12}}{\Delta_2 S_{13}} + \frac{\Delta_4 S_{23}}{\Delta_2 S_{34}}\right)^{-1}, \tag{10.2}$$

$$m_3 = \left(1 + \frac{\Delta_1 S_{23}}{\Delta_3 S_{12}} + \frac{\Delta_2 S_{13}}{\Delta_3 S_{12}} + \frac{\Delta_4 S_{13}}{\Delta_3 S_{14}}\right)^{-1}, \tag{10.3}$$

and

$$m_4 = \left(1 + \frac{\Delta_1 S_{34}}{\Delta_4 S_{13}} + \frac{\Delta_2 S_{34}}{\Delta_4 S_{23}} + \frac{\Delta_3 S_{14}}{\Delta_4 S_{13}}\right)^{-1}. \tag{10.4}$$

Note that we are free to substitute a different S_{ij}/S_{ik} in each term from the consistency equations.

10.1 Face 1: First isosceles case

Definition 19. *Type-1 isosceles configurations* have $r_{12} = 1$, $1 \le r_{13} = r_{23} \le \sqrt{2+\sqrt{3}}$, and $1/\sqrt{3} \le r_{14} = r_{24} \le 1$.

The first face of T_b, where $\phi = \pi/4$, parameterizes the type-1 configurations. On the interior of this face $\mathcal{F}$ is already defined so no extension is necessary. The symmetry of these configurations implies that $\mathcal{F}_1 = \mathcal{F}_2$.

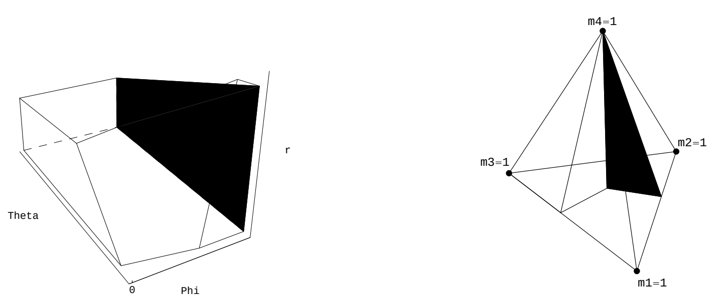


Figure 10.3: Isosceles type-1 face and part of its image in $\mathcal{M}$ (schematic)

10.2 Face 2: Second isosceles case

Definition 20. *Type-2 isosceles configurations* are configurations with $r_{12} = r_{23} = 1 < r_{13} < \sqrt{2}$ and $\frac{r_{13}}{2} < r_{14} = r_{34} < r_C = \frac{1}{\sqrt{4-r_{13}^2}}$.

Just as for the type-1 configurations, we already have the z_{i4} defined on the interior of the second face, where $\phi = 0$, which corresponds to type-2 configurations. All of the components of $\mathcal{F}$ are continuous and well-defined on interior points of T_2.

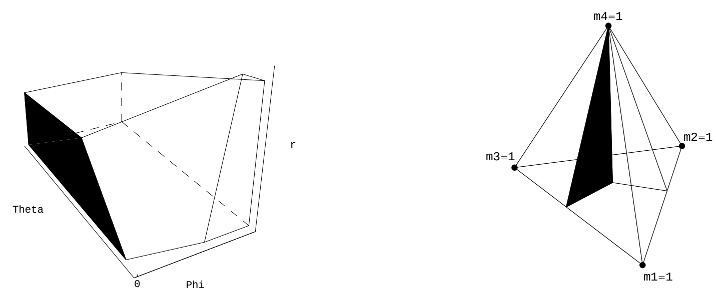


Figure 10.4: Type-2 isosceles face T_{2b} and part of its image (schematic)

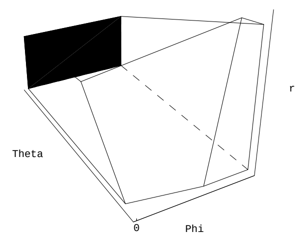
10.3 Face 3: Outer equilateral triangle

The third face of the boundary of T_b consists of points with $\theta = \pi/2$, that is points $p_0 = (r_0, 0, c)$ in coordinates (r_0, a, c). These configurations have an equilateral outer triangle ($r_{23} = r_{13} = 1$). As the following lemma shows, we can smoothly extend the z_{i4} functions so that the fourth point is at the circumcenter of the outer triangle.

Lemma 13. *For $p \in T_b$, $\lim_{p \to p_0} z_{i4} = 1/\sqrt{3}$.*

Proof. We simply examine the consistency equations (4.2) and note that at p_0 we have $S_{12} = S_{13} = S_{23} \neq 0$, which implies that $S_{14} = S_{24} = S_{34}$. That, in turn, implies that the inner distances z_{i4} must become equal as $p \to p_0$, which means they will be equal to the circumradius $1/\sqrt{3}$. $\square$

It follows that the outer masses must be equal ($\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3$), and m_4/m_1 is determined by the value of r_0. Since the limit is independent of c the blowup is somewhat redundant, but this does not cause any problems.



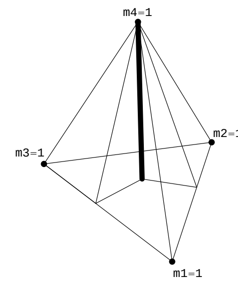


Figure 10.5: Outer equilateral face of T_b and its image in $\mathcal{M}$ (schematic)

Figure 10.5 shows the third face in our schematic of T_b, and part of its image under the extended mass map $\mathcal{F}$. The redundancy of the blowup is seen by the one-dimensionality of the image.

10.4 Face 4: Border configuration boundary

For points in the interior of the fourth face, where $r_0 = r_I$, we have $D_2 = 0$ and $z_{34}+z_{14} = r_{13}$ by definition. For all these points we can extend m_2 to be zero. The other components of $\mathcal{F}$ extend continuously as well, with the simplification that $\sigma_{2i} = 0$ for $i \in \{1, 3, 4\}$, since the strict inequalities $z_{34} < z_{14} < z_{24} < r_0 < 1 < r_{23} < r_{13}$, $D_1 > 0$, $D_3 > 0$, and $D_4 > 0$ prevent any terms of $\mathcal{F}$ from becoming singular.

Figure 10.6 schematically depicts the points of the fourth face along with the part of their image in $\mathcal{M}_c$. It should be pointed out that this face of T_b is not actually flat in the (r, θ, ϕ) coordinates. It is a smooth surface $\theta_i(r, \phi)$, and from numerical studies it appears to be fairly close to a flat face (cf. Figure 10.2). Its size has also been somewhat exaggerated for clarity.

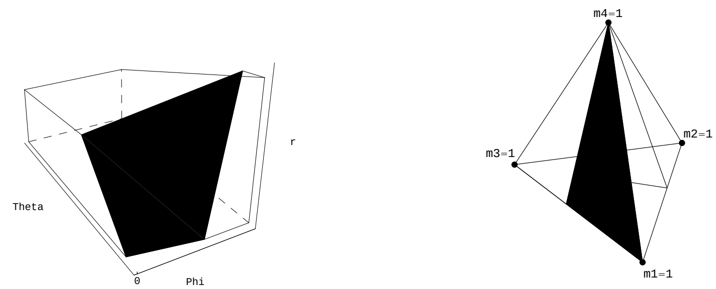


Figure 10.6: Border configuration face of T_b and part of its image in $\mathcal{M}$ (schematic)

10.5 Face 5: $r = 0$ *boundary*

For the interior of the fifth face, where $r = 0$, let $p_0 = (0, a, b)$ in coordinates (r_0, a, b). This is part of the blowup of the point $(1, 1, 1)$ in the original coordinates (r_{23}, r_{13}, r_0). These points are shown in Figure 10.7 along with the part of their image under $\mathcal{F}$ which is in $\mathcal{M}_c$.

Lemma 14. *The functions z_{14}, z_{24}, and z_{34} extend continuously to the interior of the fifth face of T_b.*

Proof. The proof of Theorem 7 is sufficient for this lemma once we change into the (r_0, a, b) coordinates. Then $F_{14} = R_0 + (1+a)(R_{24} - R_0)/(1+b)$ and $F_{34} = R_0 + (1+a)(R_{24} - R_0)$. These functions are well behaved on the interior of the fifth face, and the proof of Theorem 7 goes through, *mutatis mutandis.* In fact the construction also goes through for the interior of edges T_{15}, T_{25}, and T_{35}, where $b = 0$, $b = a$, and $a = b = 0$ respectively. $\square$

10.6 Face 6: $R_0 = R_C$ *boundary*

The case of the interior of the sixth face, where $R_0 = R_C$, is particularly simple. Lemma 4 implies that the z_{i4} all continuously extend to be equal to R_C. This in turn implies that

Lemma 15. *For $p_0 = (r_C, r_{13}, r_{23}) \in T_b$ and a point $p \in T_b$,*

$$\lim_{p \to p_0} m_4 = 1.$$

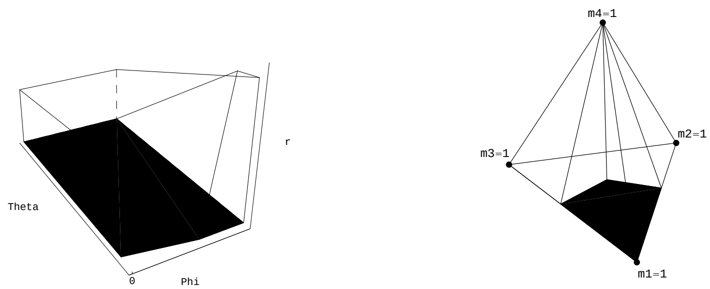


Figure 10.7: $r = 0$ face of T_b and part of its image in $\mathcal{M}$ (schematic)

Proof. Since $S_{14} = S_{24} = S_{34} = 0$, the result follows immediately from (10.4) since the other quantities are non-zero. □

Of course, Lemma 15 implies that the other masses must be zero, i.e. $\lim_{p \to p_0} m_i = 0$ for $i = 1, 2, 3$.

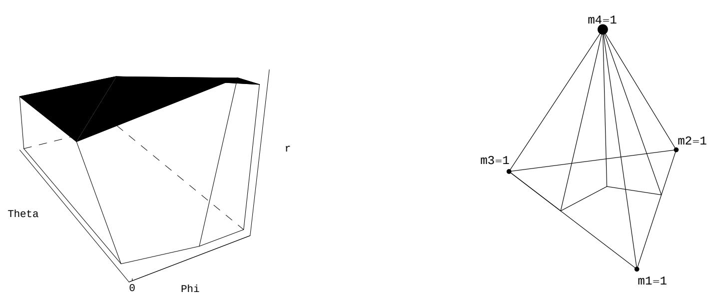


Figure 10.8: $R_0 = R_C$ face of T_b and its image in $\mathcal{M}$

Figure 10.8 shows the sixth face shaded in our schematic of T_b, along with part of its image (the point $m_4 = 1$ in $\mathcal{M}$).

10.7 Face 7: $R_0 = 1$ *boundary*

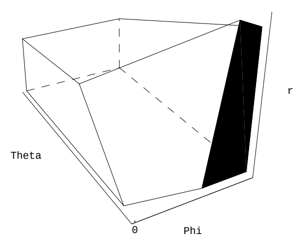
For the interior of the seventh face, we examine points $p_0 = (r_{23}, r_{13}, 1)$ in our original coordinates (r_{23}, r_{13}, r_0), where $r_{23}^2 \geq r_{13}^2 - r_{13} + 1$. Lemma 3 implies that the z_{i4} extend smoothly to this part of T_b, with $z_{14} = z_{24} = 1$ and

$$z_{34} = 1 + r_{23}^2 + r_{13}^2 - \sqrt{3[2(r_{13}^2 + r_{23}^2) - 1 - (r_{13}^2 - r_{23}^2)^2]}.$$

We can compute that $\lim_{p \to p_0} m_3 = 0$ from (10.3) and by noting that $\lim_{p \to p_0} S_{14} = \lim_{p \to p_0} S_{24} = \lim_{p \to p_0} S_{12} = 0$ while the other S_{ij} and the Δ_i stay non-zero.

The other masses have well-defined limits as $p \to p_0$ that do not simplify much except for the disappearance of the m_3/m_i term.

In Figure 10.9 this face is depicted along with part of its image in $\mathcal{M}$.



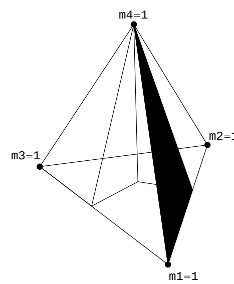


Figure 10.9: $R_0 = 1$ face of T_b and part of its image in $\mathcal{M}$

10.8 Edges

In this section we see what the previous extensions of $\mathcal{F}$ to the faces imply for the extension to the edges.

As with the interior of the sixth face, we can extend $\mathcal{F}$ to all the edges T_{i6} with the constant value of $(0, 0, 0, 1)$. The continuity of this extension is easy to check, since $\lim_{p \to p_0} z_{i4} = r_C$ for any $p \in T_b$ and $p_0 \in T_{i6}$.

The extension of $\mathcal{F}$ to edges T_{13} and T_{23} is also relatively simple. To preserve continuity with our extension to T_3 we must let $\mathcal{F}(p_0) = (m_1, m_1, m_1, 1 - 3m_1)$, where $p_0 \in T_{23}$ or p_{13} and $m_1 = (3 + \frac{3(R_0 - 1)}{R_C - R_0})^{-1}$.

As noted in the proof of Lemma 14, we can extend the functions z_{i4} continuously to the edges T_{15}, T_{25}, and T_{35}. We can then extend $\mathcal{F}$ to these edges without difficulty, as any singular expressions can be avoided by an appropriate choice of the σ_{ij}. On T_{15} we then have

$$\mathcal{F}_1 = \mathcal{F}_2 = \frac{z_{34}^{-2} - z_{34}}{2(z_{34}^{-2} - z_{34}) + \sqrt{4z_{14}^2 - 1}(z_{14}^{-3} - 1)}$$

where $z_{34} = \sqrt{1/2 + z_{14}^2 - \sqrt{3(4z_{14}^2 - 1)}}$. Since $\mathcal{F}_4 = 0$ on this edge, $\mathcal{F}_3 = 1 - 2\mathcal{F}_1$. The formulae for T_{25} are identical up to the substitution of z_{24} for z_{34}. On the edge T_{35}, $\mathcal{F}$ extends continuously to the value $(1/3, 1/3, 1/3, 0)$.

For points on T_{17}, we can apply Lemma 3 to see that we can continuously extend the z_{i4} by letting $z_{14} = z_{24} = 1$ and $z_{34} = \sqrt{1/2 + r_{13}^2 - \sqrt{3(4r_{13}^2 - 1)}}$. $\mathcal{F}$ continuously extends to be $(m_1, m_1, 0, 1-2m_1)$, in which the expression $m_1 = (2+\frac{D_4(1-R_{13})}{D_1(Z_{34}-1)})^{-1}$ does not appreciably simplify further.

By the definition of r_I we know that the z_{i4} extend continuously to a point $p_0 = T_{24}$ with $z_{34} = z_{14} = r_{13}/2$ and $z_{24} = \sqrt{1 - r_{13}^2/4}$. Recall that $p_0 = (r_{13}, r_{13}, r_I)$ in coordinates (r_{23}, r_{13}, r_0). As on the rest of the face T_4, since $\Delta_2 = 0$ we extend by setting $\mathcal{F}_2(p_0) = 0$. There are no other singular terms in the $\mathcal{F}_i$, and we find that $\mathcal{F}_1(p_0) = \mathcal{F}_3(p_0) = (2 + 2\frac{R_I - R_{13}}{8R_{13} - R_I})^{-1}$ and $\mathcal{F}_4(p_0) = 1 - 2\mathcal{F}_1(p_0)$.

The points $p_0 \in T_{47}$ consist of points $((r_{13}^2 - r_{13} + 1), r_{13}, 1)$ in coordinates (r_{23}, r_{13}, r_0). By considering the behavior of the z_{i4} on the faces T_4 and T_7 it is clear that we can extend $z_{14}(p_0) = z_{24}(p_0) = 1$ and $z_{34}(p_0) = r_{13} - 1$. So just as for the interior of T_7 at p_0 we have $S_{14} = S_{24} = S_{12} = 0$.

For points on the edge T_{45} we know that the z_{ij} extend continuously by the arguments in Chapter 8. These points have the form $(1, d_i(c), c)$ in coordinates (r_{13}, d, c). Recall that for these points $D_2 = 0$. F extends continuously to these points with $\mathcal{F}_2 = \mathcal{F}_4 = 0$, $\mathcal{F}_4 = 1 - \mathcal{F}_1$, and $\mathcal{F}_1 = (1 + \frac{(\sqrt{3}/4 - D_1)(Y_{14}-1)}{(D_1)(W_{34}-1)})^{-1}$. In the expression for $\mathcal{F}_1$ note that Y_{14} is a function of c and $W_{34} = (1 - Y_{14}^{-1/3})^{-3}$.

The edge T_{57} consists of points $p_0 = (1, 0, c)$ in coordinates (r_{13}, d, c). Note that for this edge the coordinate c is in the interval $(1/2, 1)$. Here the extension is a bit trickier. We begin with the following lemma.

Lemma 16. *For $p \in T_b$ and $p_0 \in T_{57}$, $\lim_{p\to p_0} z_{14} = \lim_{p\to p_0} z_{24} = 1$ and $\lim_{p\to p_0} z_{34} = 0$.*

Proof. If we let $1+\epsilon_{13} = r_{13}$ then we have $r_{23} = 1+c\epsilon_{13}+O(\epsilon_{13}^2)$. In terms of our coordinates (r_{13}, d, c) we have $r_0 = 1 - d\epsilon_{13} + O(\epsilon_{13}^2)$, $F_{14} = 1 + (1 + d)(R_{24} - 1)/(d + c) + O(\epsilon_{13})$ and $F_{34} = 1 + (1 + d)(R_{24} - 1)/d + O(\epsilon_{13})$. Recall that F_{14} and F_{34} are used to define the z_{i4} in Theorem 7.

Using these expansions we compute that for $r_{24} = (1 - 5d)$ we have

$$P(1, r_{23}, F_{34}^{-1/3}, F_{14}^{-1/3}, r_{13}, 1 - 5d) = -6d + O(d)O(\epsilon_{13}) + O(d^2),$$

which is negative for sufficiently small d and ϵ_{13}. Recall from the proof of Theorem 7 that at the endpoint where $r_{24} = r_0$ that $P > 0$. Thus we know that for sufficiently small d and ϵ_{13}, z_{24} is between $(1 - d\epsilon_{13})$ and $(1 - 5d)$, which means that $\lim_{p\to p_0} z_{24} = 1$. From the expression for F_{14} we can see that this implies that $\lim_{p\to p_0} z_{14} = 1$. Finally, the fact that $\lim_{p\to p_0} z_{34} = 0$ follows from planarity. □

With Lemma 16 in hand, we now examine the behavior of z_{i4} near p_0 in more detail. We will use the same expansions for r_{13} and r_{23} as in Lemma 16. From the relation $S_{13}/S_{23} = S_{14}/S_{24}$ we find that if $z_{14} = 1 + \epsilon_{14}$ then $z_{24} = 1 + c\epsilon_{14} + O(d)$. Using these expansions we find that

$$\lim_{p\to p_0} \sigma_{21} = \frac{(2c - 1)}{c(2 - c)}, \tag{10.5}$$

after which it is easy to compute that $\mathcal{F}$ extends continuously to T_{57} with values

$$\lim_{p\to p_0} m_1 = c(2 - c)/(4c - c^2 - 1),$$

$$\lim_{p\to p_0} m_2 = (2c - 1)/(4c - c^2 - 1),$$

$$\lim_{p\to p_0} m_3 = 0,$$

and

$$\lim_{p\to p_0} m_4 = 0.$$

10.9 Vertices

We can complete our continuous extension of $\mathcal{F}$ by studying its behavior near the vertices.

It is easy to see from previous sections that we can let $\mathcal{F}(T_{ij6}) = (0,0,0,1)$. It is also trivial to determine that we can let $\mathcal{F}(T_{235}) = \mathcal{F}(T_{135}) = (1/3, 1/3, 1/3, 0)$.

The vertex T_{245} can be described as $(1, \frac{21\sqrt{3}-19}{208}, 0)$ in coordinates (r_{13}, d, c). To see that the z_{14} have a continuous extension to this point we must again revisit the functions F_{14} and F_{34} from the proof of Theorem 7.

Lemma 17. *For $p \in T_b$ and $p_0 = T_{245}$, $\lim_{p\to p_0} z_{14} = \lim_{p\to p_0} z_{34} = 1/2$ and $\lim_{p\to p_0} z_{24} = \sqrt{3}/2$.*

Proof. Let $p = (r_{13}, d, c)$, with $r_{13} = 1 + \epsilon_{13}$ and $d = \frac{21\sqrt{3}-19}{208} + \epsilon_d = d_0 + \epsilon_d$. Then both F_{14} and F_{34} are of the form $1 + (1 + 1/d_0)(R_{24} - 1) + O(\epsilon_{13}) + O(\epsilon_d) + O(c)$. We can compute that

$$P(1, 1 + O(\epsilon_{13}), F_{34}^{-1/3}, F_{14}^{-1/3}, 1 + \epsilon_{13}, r_{24})|_{r_{24}=\sqrt{3}/2} = O(\epsilon_{13}) + O(\epsilon_d) + O(c).$$

By Lemma 5 this means that $\lim_{p\to p_0} z_{24} = \sqrt{3}/2$. It is then easy to compute from the expressions for F_{14} and F_{34} the rest of the lemma. □

With Lemma 17 in hand, we can continuously extend $\mathcal{F}$ to this vertex by $\mathcal{F}(T_{245}) = (1/2, 0, 1/2, 0)$.

The extension of $\mathcal{F}$ to the vertices T_{157} and T_{457} is covered by the arguments for the edge T_{57}. The point T_{157} is $(1, 0, 1)$ in coordinates (r_{13}, d, c), and the formulae for $\mathcal{F}(T_{57})$ specialize to $\mathcal{F}(T_{157}) = (1/2, 1/2, 0, 0)$. Likewise we can let $\mathcal{F}(T_{457}) = (1, 0, 0, 0)$.

Proof of Theorem 11. The preceding sections (10.1 - 10.9) have shown that $\mathcal{F}$ extends continuously to the boundary of T_b by examining its behavior on every boundary piece. □

Chapter 11

SURJECTIVITY OF THE MASS MAP

In this chapter we examine the degree of the extended map $\mathcal{F}$ on the boundary of its domain, in order to prove the following theorem.

Theorem 12. *Every point in $\mathcal{M}_c$ has a preimage of $\mathcal{F}$ in T_b.*

Definition 21. Let $\mathcal{M}_1$ denote the set $\{(m_1, m_2, m_3, m_4)| \sum m_i = 1,\ m_i \geq 0,\ m_1 = m_2, m_1 \geq m_3\}$. Let $\widetilde{\mathcal{M}_1}$ denote the set $\{(m_1, m_2, m_3, m_4)| \sum m_i = 1,\ m_i \geq 0,\ m_1 = m_2\}$.

Note that $\partial\mathcal{M}_1$ is the union of the segments $\{(m_1, m_1, 1-2m_1, 0) \mid \frac{1}{3} \leq m_1 \leq \frac{1}{2}\}$, $\{(m_1, m_1, 0, 1-2m_1) \mid 0 \leq m_1 \leq \frac{1}{2}\}$, and $\{(m_1, m_1, m_1, 1-3m_1) \mid 0 \leq m_1 \leq \frac{1}{3}\}$.

Lemma 18. *$\mathcal{F}_4$ restricts to a monotonic function of r_0 on T_{13}, whose image is the segment $[0, 1]$.*

Proof. From section 10.8 we know that the image of T_{13} under $\mathcal{F}$ is contained in the segment $\{(m_1, m_1, m_1, 1-3m_1) \mid 0 \leq m_1 \leq \frac{1}{3}\}$. From section 10.9 we know that $\mathcal{F}_4(T_{135}) = 0$ and $\mathcal{F}_4(T_{136}) = 1$. Recall from section 10.8 that a point p in T_{13} has the form $(r_0, 0, 1)$ in coordinates (r_0, a, c) and that $\mathcal{F}_4(p) = 1 - 3(3 + \frac{3(R_0-1)}{3\sqrt{3}-R_0})^{-1}$. We can simplify $\mathcal{F}_4(p)$ to the form $\frac{R_0-1}{3\sqrt{3}-1}$, which is clearly monotonic in r_0. $\square$

Lemma 19. *The degree of the map $\mathcal{F}|_{\partial T_1}$ from ∂T_1 to ∂M_1 is ± 1.*

Proof. This is almost immediate from Lemma 18. We need only note that the images of edges T_{17}, T_{15}, and T_{16} are disjoint from the interior of the mass core $\{(m_1, m_1, m_1, 1-3m_1) \mid 0 \leq m_1 \leq \frac{1}{3}\}$. Thus for q in the mass core, there is only one preimage $\mathcal{F}|_{\partial T_1}^{-1}(q)$ and by Lemma 18 q must be a regular value of $\mathcal{F}|_{\partial T_1}$. This implies that degree of $\mathcal{F}|_{\partial T_1}$ is ± 1 (cf. Hirsch [Hr]). $\square$

Lemma 20. *The degree of the mass map from the boundary of T_b to the boundary of $\mathcal{M}$ is ± 1.*

Proof. In previous sections we have seen that the image under $\mathcal{F}$ of the various faces of T_b are each contained in a set of the form $L_i \cap \partial\widetilde{\mathcal{M}}_c$ where L_i is some proper linear subspace of $\mathbb{R}^4$. This means that the following diagram commutes,

$$\begin{tikzcd} \partial T_b \arrow[r, "\mathcal{F}"'] \arrow[d, "Q_T"'] & \partial\widetilde{\mathcal{M}}_c \arrow[d, "Q_M"'] \\ T_1/\partial T_1 \arrow[r, "\tilde{\mathcal{F}}"] & \widetilde{\mathcal{M}}_1/\partial\mathcal{M}_1 \end{tikzcd}$$

in which Q_T is the quotient map that collapses the complement of the interior of T_1 to a point, and Q_M is the quotient map that collapses $\partial\mathcal{M}_1$ and the complement of $\widetilde{\mathcal{M}}_1$ to a point.

Note that the sequence of maps

$$\partial\mathcal{M}_1 \overset{i}{\hookrightarrow} \widetilde{\mathcal{M}}_1 \to \widetilde{\mathcal{M}}_1/\partial\mathcal{M}_1 \tag{11.1}$$

induces a long exact sequence of homotopy groups which collapses to isomorphisms between $\pi_{n+1}(\widetilde{\mathcal{M}}_1/\partial\mathcal{M}_1)$ and $\pi_n(\partial\mathcal{M}_1)$. In particular the isomorphism between $\pi_2(\widetilde{\mathcal{M}}_1/\partial\mathcal{M}_1)$ and $\pi_1(\partial\mathcal{M}_1)$ implies that $\deg(\tilde{\mathcal{F}}) = \deg(\mathcal{F}|_{\partial T_1}) = \pm 1$. Since the quotient maps Q_T and Q_M are homotopic to the identity, $\deg(\mathcal{F}) = \deg(\tilde{\mathcal{F}}) = \pm 1$ as well. ☐

Proof of Theorem 12. This follows immediately from Lemma 20 and Lemma 11. ☐

Chapter 12

NUMERICAL EXPERIMENTS

Although none of the proofs presented here rely on upon numerical evidence, many numerical experiments were helpful in clarifying how some of the faces fit together. Computing the concave central configurations for a fixed outer triangle was perhaps the most useful of these experiments, and so four representative examples are shown in Figure 12.1. The curve of fourth points is parameterized by r_0. They always begin at the circumcenter of the triangle ($r_0 = r_C$) and end at either $r_0 = 1$ or $r_0 = r_I$ depending on whether or not the third point is above or below the diagonal in Q, respectively.

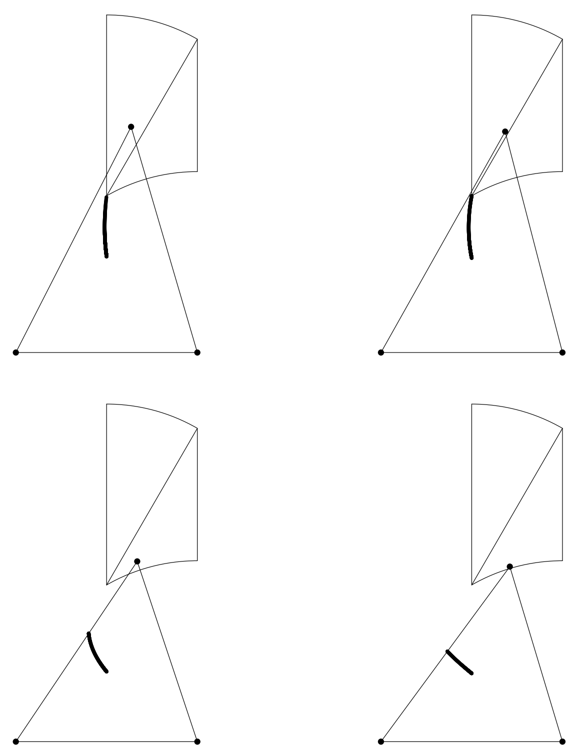


Figure 12.1: Curves of points for a given outer triangle

Chapter 13

CONCLUSION

Proof of Theorem 1. In chapter 6 a canonical representative for each equivalence class of concave central configurations was defined. Then Theorem 8 showed that the set of such representatives of concave central configurations is homeomorphic to a three-dimensional ball (including only part of its boundary). In chapter 10 the map $\mathcal{F}$ was then shown to extend continuously to the boundary of that set after blowing up some of the points on the boundary. Finally in chapter 11 the extended map restricted to the boundary is shown to have non-zero degree onto its image, which implies that the map is surjective onto $\mathcal{M}$ by Lemma 11 in chapter 9. □

Combined with the previously known information on central configurations of the four-body problem we have the following conclusion:

Theorem 13. *For any four positive masses $(m_1, m_2, m_3, m_4) \in \mathbb{R}^4_+$ there are the following equivalence classes of central configurations: exactly one spatial class [Sa1], exactly 12 collinear classes [Mu], at least one convex class [MB], no noncollinear classes with three collinear points [ZY], and at least one concave class.*

We can also observe a couple of corollaries to the proof of Theorem 1.

Corollary 1. *For four masses where no pair of masses are equal there are at least eight concave central configurations.*

Proof. Given masses m_a, m_b, m_c, and m_d, we can choose any of them to be m_4. Then in order to be in the image of our map $\mathcal{F}$ we must choose the greatest of the remaining masses to be m_1. Finally, we are left free to choose either of the remaining two masses to be m_2. These eight choices correspond to eight distinct points in $\mathcal{M}_c$, which have eight distinct preimages in T_b. □

Corollary 2. *There exist concave central configurations for any four masses with* $m_1 > m_2 = m_3 = m_4 > 0$ *such that none of the interparticle distances* r_{ij} *are equal.*

Proof. This is immediate from the fact that $\mathcal{M}_c$ contains the segment where $m_1 > m_2 = m_3 = m_4$, and the preimages of this segment with respect to $\mathcal{F}$ will have no equal interparticle distances. □

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