Course Summary
Dynamical Systems
Math 5260
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December 10, 2013

Dynamical Systems: The study of systems which evolve in time.

1. Continuous time (differential equations): $\dot{x} = f_c(x), \;\; x \in \Re^n, c \in \Re^k$.

2. Discrete time (maps): $x_{n+1} = f_c(x_n), \;\; x \in \Re^n$ or $\mathbf{C}^n$, $c \in \Re^k$ or $\mathbf{C}^n$.

General Goal: Short version:

1. **Compute** explicit general solutions when possible (usually only for linear, or certain 1D DE's).

2. For individual dynamical systems, determine **qualitative features** of solutions (**phase portraits**). Focus on *limiting behavior* (α- and ω-limit sets) and how this limiting behavior depends on initial conditions.

3. For families of dynamical systems, divide the parameter space up into equivalence classes. Determine when and how these qualitative features change as parameters are varied to go from one equivalence class to another (**bifurcation**).

4. Recognize indicators of chaos.

5. Know simplest examples of chaos in various settings.

Long version:

1. **Compute** explicit general solutions solutions in terms of time (t or n) and initial conditions (x_0) for *linear* systems: $\dot{x} = Ax$ has general solution $x(t) = e^{At}x_0$; $x_{n+1} = Ax_n$ has general solution $x_n = A^n x_0$. This is most easily done for certain *canaonical* matrices. In 2D, these canonical matrices are diagonal (distinct real eigenvalues), "$a, -b, b, a$" (complex eigenvalues $a \pm ib$), and "$\lambda, 1, 0, \lambda$" (repeated real eigenvalues of λ). Use polar coordinates for the complex case.

2. For individual (nonlinear) dynamical systems, determine **qualitative features** of solutions (**phase portraits**).

   (a) 1D DE's: Locate equilibria, determine arrows for stability using, for example, plot of $\dot{x}$ versus x. All solutions are monotonic. Neither periodic orbits nor chaos is possible.

(b) 2D DE's: Locate equilibria, linearize to determine stability, determine stable and unstable manifolds of all saddles, look for or rule out periodic orbits (Index theory, Poincare-Bendixon, gradient systems, ...), Sketch phase portrait, possibly using phase plane software, nullclines, ... Chaos is not possible (Poincare-Bendixon).

(c) 3D DE's: Same as 2D DE's, but use 3D software to view. Poincare-Bendixon does not apply, so **chaos** on **strange attractors** is possible. Lorenz attractor.

(d) 1D maps:

   i. orientation preserving homeomorphisms: Locate fixed points, determine arrows for stability. Neither periodic solutions (other than fixed points) nor chaos is possible. (same as for 1D DE's)

   ii. orientation reversing homeomorphisms: second iterete is orientation preserving. Fixed points of the second iterate can be period-two points of the original map.

   iii. Non homeomorphisms, eg. quadratic. Analytically locate fixed points, period-two points, period-three points, ... Determine stability (chain rule). Look for invariant sets (eg. intervals, Cantor sets). **Chaos** is possible. ($x \mapsto x^2 + c$ for certain c values.)

   iv. More complicated non homeomorphisms: higher degree polynomials, rational functions (Newton's method), transcendental functions. Same as for quadratic maps, but more to keep track of.

(e) 2D maps:

   i. Homeomorphisms: Locate fixed points, linearize to determine stability, determine stable and unstable manifolds of all saddles, look for or rule out periodic orbits. Use software to iterate. **Chaos** is possible. (Henon map.)

   ii. Noninvertible: Start with ideas for homeomorphisms, but much more ... . Current area of research.

   iii. Complex 1D: Locate attracting periodic orbit(s) and their basin boundaries: Filled Julia sets (bounded orbits) and Julia sets (**Chaos**).

(f) 3D maps: ... ???????? Current area of research.

3. For **families** of dynamical systems, divide the parameter space up into equivalence classes. Determine when and how these qualitative features change as parameters are varied to go from one equivalence class to another (**bifurcation**).

   (a) 1D DE's with k parameters

   - $k = 1$: Sketch equilibrium set in phase $\times$ parameter space. Determine regions of $\dot{x}$ positive vs. negative. Locate bifurcation points by inspection or by $f_c(x) = 0, f_c'(x) = 0$. Determine stablility of equilibria (make equilibria solid or dashed curves). Add phase lines for each equivalence class. Classify bifurcations. as saddle-node (tangent), transcritical, pitchfork, or other.
   - $k = 2$: Either try to sketch the surface(s) of equilibria in the three-dimensional phase $\times$ parameter space and proceed as for $k = 1$, or divide parameter plane into equivalence classes using curves defined by $f_c(x) = 0, f_c'(x) = 0$ and eliminating x (projecting to the parameter plane). Add phase lines for each equivalence class. Classify bifurcations: codimension-one curves (same as bifurcation points for 1D), codimension-two points (lots of complicated bifurcations) One parameter cuts are sometimes useful (fix one parameter and treat as a one-parameter family).
   - $k > 2$: No obvious strategy which works best in all situations. Do calculations as much as possible with all parameters together. If difficulties arise, fix some parameters, ... , ?

   (b) 2D DE's with k parameters.

   - $k = 1$: Sketch bifurcation diagrams in 3D, or project to 2D. For, example, sketch the x_1 coordinate OR the x_2 coordinate of equilibria as a function of the parameter, whichever one seems to display more information. Or show both. Or, divide parameter line into equivalence classes using points defined by $f_c(x) = 0, \det(Df_c(x)) = 0$ (eigenvalue of zero — usually saddle-node), or $f_c(x) = 0, \text{trace}(Df_c(x)) = 0, \det(Df_c(x)) > 0$ (pure imaginary eigenvalues — usually Hopf). Use phase plane software to help determine phase portraits for each equivalence class. Determine global bifurcations (not always so easy): saddle connections, saddle-node of limit cycles, ... Note that periodic orbits can be born in Hopf bifurcations and in saddle-connection bifurcations. This could involve extensive use of phase plane software, as well as Mathematica, Maple, ... Include phase portraits for each equivalence class.

- $k = 2$: Divide parameter plane into equivalence classes. The same sets of equations as for $k = 1$ now give curves of saddle-node bifurcations (eigenvalue of zero), curves of Hopf bifurcations (pure imaginary eigenvalues), curves of saddle-connections, ... Be especially careful near where any two such curves intersect. Include phase portraits for each equivalence class. One parameter cuts are sometimes useful.
- $k > 2$: Good luck.

(c) Maps. Similar strategy to DE's. Just harder to divide into equivalence classes. Remember that for DE's stablilty is determined by whether eigenvalues are in the right or left-half plane, but stability for maps is determined by whether eigenvalues are inside or outside the unit circle. For example, a saddle-node requires an eigenvalue of one, period doubling an eigenvalue of negative one, and Hopf a pair of complex eigenvalues on the unit circle.

4. Recognize indicators of chaos:

   (a) 3D DE: Strange attractors (Lorenz attractor). Non point, non limit cycle attractor.

   (b) 1D map: Wild cobweb orbit; graph pushed through "square" implies invariant Cantor set ($x^2 + c$ for $c < -2$); filled out interval on orbit diagram ($x^2 - 2$, doubling map, tent map, V map); orbit diagram like quadratic map or logistic map.

   (c) 1D complex map: Julia sets, Mandelbrot set

   (d) 2D map: Homoclinic tangles, horseshoes

   (e) Most cases: stretching (orbits separate, SIC) and folding (orbits can come back together, allowing periodic orbits and transitive orbits)

5. Know simplest examples of chaos in various settings.

   (a) 3D differential equations: Lorenz equations with Lorenz attractor

   $$(\dot{x}, \dot{y}, \dot{z}) = (\sigma(y - x), rx - y - xz, xy - bz)$$

   Classic parameters: $(\sigma, b, r) = (10, 8/3, 28)$.

   (b) 1D noninvertible maps: $Q_c(x) = x^2 + c$

   (c) 1 complex dimension maps: $Q_c(z) = z^2 + c$

   (d) 2D maps: Henon map with Henon attractor

   $$(x, y) \to (1 - ax^2 + y, bx)$$

   Classic parameters: $(a, b) = (1.4, 0.3)$