Math 3280, Differential Equations with Linear Algebra

Test 3

B. Peckham

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Name A. K.

SHOW ALL WORK.
Please do not write on the provided Laplace Transform tables.
Indicate clearly any places where you use either the Laplace transform tables or a calculator.

Name:	________ /4
p2.	________/20+4
p3.	________/35
p4.	________/27
p5.	________/14+6
Total	________/100+10

1. (6 pts) Find the general solution to $2y'' - y' - 3y = 3e^{-2x}$.

For y_c: $2r^2 - r - 3 = 0$

$(2r - 3)(r + 1) = 0$

$r = +\frac{3}{2}, -1$

$\Rightarrow y_c = c_1 e^{\frac{3}{2}x} + c_2 e^{-x}$

Try $y_p = Ae^{-2x} \Rightarrow y_p' = -2Ae^{-2x}$, $y_p'' = 4Ae^{-2x}$ } Plug in:

$2 \cdot 4Ae^{-2x} + 2Ae^{-2x} - 3Ae^{-2x} = 3e^{-2x}$

i.e. $7A = 3 \Rightarrow A = 3/7$

$\therefore y(x) = c_1 e^{+\frac{3}{2}x} + c_2 e^{-x} + \frac{3}{7} e^{-2x}$

2. (a) (6 pts) Write down a constant coefficient linear homogeneous differential equation that has e^{3x} and $e^{3x}\cos(2x)$ as two solutions. You determine the order of the differential equation.

$e^{3x} \to r = 3$, $e^{3x}\cos(2x) \to 3 \pm 2i$

$\therefore (r-3)(r-(3+2i))(r-(3-2i))$

$= (r-3)((r-3)^2 + 4) = (r-3)(r^2 - 6r + 9 + 4)$

$= (r-3)(r^2 - 6r + 13)$

$= r^3 - 9r^2 + 31r - 39$

$\Rightarrow y''' - 9y'' + 31y' - 39y = 0$

(b) (4 pts Extra Credit) What linear differntial operator annihilates $4e^{3x} + 5e^{3x}\cos(2x)$? Hint: use part (a).

$(D-3)((D-3)^2 + 4)$

which is the same as $D^3 - 9D^2 + 31D - 39$

3. Find the form of a particular solution to the following differential equations. Do not include any terms that are part of the complementary (homogeneous) solution, and do not evaluate the "undetermined coefficients."

(a) (4 pts) $y'' + 4y = \sin(t)$ $y_p = A\sin t + B\cos t$ ✓

y_c: $r = \pm 2i$

$\to \sin 2t, \cos 2t$

(b) (4 pts) $y'' + 4y = \sin(2t)$. $y_p = At\sin 2t + Bt\cos 2t$

Same

(Since the initial guess of $y_p = A\sin 2t + B\cos 2t$, and both $\sin 2t$ and $\cos 2t$ are solutions to y_c, then change the guess to $At\sin 2t + Bt\cos 2t$.)

4. (5 pts) What is the Laplace transform of $g(t) = 3te^{2t} - u(t-4)$? ~~(You~~ may use the tables. You need not simplify your answer.) $+e^{t}\cos 3t$

$$G(s) = 3\frac{1}{(s-2)^2} - \frac{e^{-4s}}{s} + \frac{s-1}{(s-1)^2+3^2}$$

5. (8 pts) Compute the Laplace transform of $f(t) = u(t-2)$ **directly from the definition of the Laplace transform** (not the tables). ($u(t)$ is the unit step function.)

$$\mathcal{L}\{u(t-2)\}(s) = \int_0^\infty e^{-st}u(t-2)\,dt = \int_2^\infty e^{-st}\cdot 1\,dt = \frac{e^{-st}}{-s}\Big|_2^\infty = 0 - \frac{e^{-s\cdot 2}}{-s} = \frac{e^{-2s}}{s}$$

6. (6 pts) Define $f(t) = \begin{cases} t^2 & 0 \le t < 2 \\ \cos(2t) & 2 \le t < 5 \\ t & 5 \le t. \end{cases}$ Use step functions to write $f(t)$ as a single line formula.

$$f(t) = u(t-0)\,t^2 + u(t-2)(\cos 2t - t^2) + u(t-5)(t - \cos 2t)$$

7. (10 pts) Solve using the method of Laplace transforms: $y'(t) - 4y(t) = 3e^{5t}$, $y(0) = 2$.

$$(sY(s) - y(0)) - 4Y(s) = \frac{3}{s-5}$$

$$\Rightarrow Y(s) = \frac{\frac{3}{s-5} + 2}{s-4} = \frac{3}{(s-5)(s-4)} + \frac{2(s-5)}{(s-5)(s-4)} = \frac{A}{s-5} + \frac{B}{s-4} = \frac{A(s-4)+B(s-5)}{(s-5)(s-4)}$$

where $3 + 2s - 10 = 2s - 7 = A(s-4) + B(s-5)$ $\quad = 3\frac{1}{s-5} - \frac{1}{s-4}$

$s=4:\ 1 = B(-1) \Rightarrow B = -1$ $\quad \therefore\ y(t) = 3e^{5t} - e^{4t}$

$s=5:\ 3 = A$

8. (6 pts) Compute the Laplace transform of the solution of the initial value problem:

$$y''' + 3y'' + 0y' + 2y = 0; \quad y(0) = 0, y'(0) = 3, y''(0) = 0.$$

(Find only $Y(s)$, not $y(t)$.) Write your answer as a polynomial (in s) over a polynomial. You need not simplify your answer.

$$\left(s^3Y(s) - s^2y(0) - s\underset{3}{y'(0)} - y''(0)\right) + 3\left(s^2Y(s) - sy(0) - \underset{3}{y'(0)}\right) + 0 + 2Y(s) = 0$$

i.e., $Y(s)(s^3 + 3s^2 + 2) = 3s + 9$

$$\Rightarrow Y(s) = \frac{3s+9}{s^3+3s^2+2}$$

9. (8 pts) Find the inverse Laplace transform of $G(s) = \cancel{e^{-2s}\frac{3s-4}{s^2+4s}}$ $\frac{e^{-2s}(3s-4)}{(s+4)^2} = e^{-2s} \cdot F(s)$

Let $F(s) = \frac{3s-4}{(s+4)^2} = \frac{(s+4)A}{(s+4)(s+4)} + \frac{B}{(s+4)^2} = \frac{3}{s+4} - \frac{16}{(s+4)^2}$

where $3s-4 = A(s+4) + B$

$s=-4: \; -16 = B$

$s: \; 3 = A$

$\Rightarrow f(t) = 3e^{-4t} - 16te^{-4t}$

$\therefore \; g(t) = u(t-2)\, f(t-2)$

$= u(t-2)\left(3e^{-4(t-2)} - 16(t-2)\, e^{-4(t-2)}\right)$

10. (7 pts) Write the differential equation $y'' - 6y' - 2y = \cos(2t)$ with initial conditions $y(0) = 3, y'(0) = 4$ as an equivalent system of first order differential equations. Write the system in vector form: $\vec{\dot{x}} = A\vec{x} + \mathbf{b}$. Write the initial conditions in vector form as well.

let $v = y'$, then the d.e. becomes: $v' - 6v - 2y = \cos(2t)$

together: $y' = 0y + 1v$

$v' = 2y + 6v + \cos(2t)$

$\vec{x}(0) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

So if $\vec{x} = \begin{pmatrix} y \\ v \end{pmatrix}$, then $\dot{\vec{x}} = \begin{pmatrix} 0 & 1 \\ 2 & 6 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ \cos(2t) \end{pmatrix}$ ✓

11. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$.

(a) (8 pts) For this matrix A, find all eigenvalues and a corresponding eigenvector for each eigenvalue. Upper $\Delta \Rightarrow \lambda = 1, -2$

Space

For $\lambda = 1$ $(A - \lambda I)\vec{v} = \left(\begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)\vec{v} = \begin{pmatrix} 1-1 & 2 \\ 0 & -2-1 \end{pmatrix}\vec{v} = \begin{pmatrix} 0 & 2 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{matrix} 0 \\ 0 \end{matrix} \Rightarrow v_2 = 0$

$\Rightarrow \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda = -2$ $\begin{pmatrix} 1+2 & 2 \\ 0 & -2+2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow 3v_1 + 2v_2 = 0 \Rightarrow \vec{v} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

(b) (4 pts) Use part(a) to find the general solution to $\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \mathbf{x}$.

$\vec{x}(t) = c_1 e^{t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

12. (6pts) Let $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$. Find any one nonzero eigenvector for A. You may use the fact that one eigenvalue of A turns out to be $1 + 2i$.

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 4 \Rightarrow \lambda = 1 \pm 2i$$

For $\lambda = 1+2i$: $\begin{pmatrix} 1-(1+2i) & -2 \\ 2 & 1-(1+2i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -2i\, v_1 - 2 v_2 = 0$

$\Rightarrow -i v_1 - 1 v_2 = 0$

$\Rightarrow v_2 = -i v_1$

So if $v_1 = 1$, $v_2 = -i$: $\begin{pmatrix} 1 \\ -i \end{pmatrix}$

(Or any complex multiple of this, like $\begin{pmatrix} i \\ 1 \end{pmatrix}$)

13. Consider the system of differential equations

$$\dot{x} = x - x^2 - xy, \quad \dot{y} = 2 - y$$

↑ space ↓

(a) (4 pts) Find two equilibrium points.

$x - x^2 - xy = x(1 - x - y) = 0 \Rightarrow x(1-x-2) = 0$ or $x(-1-x) = 0$

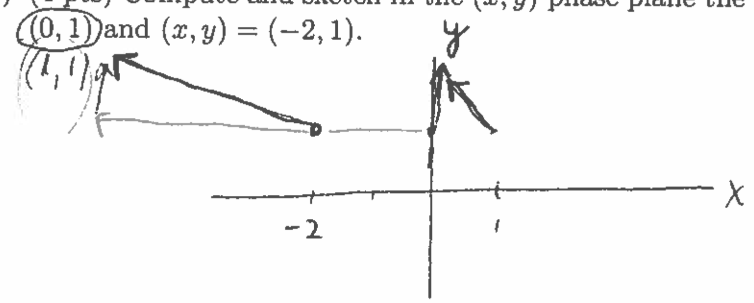
$\Rightarrow x = 0, -1$

$2 - y = 0 \Rightarrow y = 2$

$\therefore$ Eq pts: $(0,2)$, $(-1,2)$

(b) (4 pts) Compute and sketch in the (x, y) phase plane the two velocity vectors at $(x, y) =$ $(0, 1)$ and $(x, y) = (-2, 1)$.

$(1,1)$



14. (Extra credit 6 pts) **Use the definition of the Laplace transform** to show that if $F(s)$ is the Laplace transform of $f(t)$, then the transform of $f'(t)$ is $sF(s) - f(0)$.

$$\mathcal{L}\{f'(t)\}(s) = \int_0^\infty e^{-st} f'(t)\,dt = uv\Big|_0^\infty - \int_0^\infty v\,du = e^{-st} f(t)\Big|_0^\infty - \int_0^\infty e^{-st}(-s) f(t)\,dt$$

let $u = e^{-st}$ $\quad du = e^{-st}(-s)\,dt$

$dv = f'(t)\,dt \Rightarrow v = f(t)$

$$= 0 - f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= sF(s) - f(0)$$