Math 3280 Differential Equations with Linear Algebra
Lab #7: Resonant Forcing of a Spring-mass system

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Directions: Turn in a written lab report dealing with the tasks below. You may decide what tasks to do with Mathematica and which to do by hand. Be sure to include in your writeup the items in the "Lab Procedures and Guidelines" handout. Previous Lab handouts may be useful. Material in Section 5.1, 5.3, 5.4 and especially 5.6 of Edwards and Penny may also be of help.

Grading for each problem: Goals (G) 2pts, Procedures (Pro) 2pts, Presentation and Organization (P+O) 4pts, Conclusions (C) 4pts, Tasks enumerated below 18pts. Total 30pts.
The motion of a mass on a spring with an external forcing function is determined by the model differential equation:

$$25\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 50x = F_0\cos(\gamma t) \tag{1}$$

The general solution to (1) turns out to be:

$$x(t) = c_1 e^{\frac{-1}{5}t}\cos(\frac{7}{5}t) + c_2 e^{\frac{-1}{5}t}\sin(\frac{7}{5}t) + \frac{F_0}{100 - 96\gamma^2 + 25\gamma^4}[(2-\gamma^2)\cos(\gamma t) + \frac{2}{5}\gamma\sin(\gamma t)] \tag{2}$$

1. (5pts) Check that the solution given in equation (2) is actually the general solution. That is, identify "x_1, x_2, and x_p" and check that they are solutions to the "appropriate" differential equations. (You should know how to obtain the general solution even though you are not required to "solve" from scratch for this lab.) Check to see that the Wronskian determinant of x_1 and x_2 is not "identically zero." What does this guarantee about $\{x_1(t), x_2(t)\}$? Why is the "particular solution" the most important part "in the long run"?

2. Resonant behavior:

   (a) (5pts) Numerical Solutions. Find a numerical solution to equation (1) using NDSolve in *Mathematica*. Use $F_0 = 4$, $\gamma = 1$ and initial conditions $x(0) = -1$ and $x'(0) = 1$. Plot the solution. Then use the Manipulate command to vary γ in the NDSolve and Plot commands together. Estimate a value of γ which makes the output amplitude the largest.

   (b) (3pts) Show that the particular solution in equation (2) above can be rewritten in the form $x_p(t) = F_0 M(\gamma)\sin(\gamma t + \theta)$, where

   $$M(\gamma) = \frac{\sqrt{(2-\gamma^2)^2 + (\frac{2\gamma}{5})^2}}{100 - 96\gamma^2 + 25\gamma^4}$$

   (Hint: If $A\cos(\gamma t) + B\sin(\gamma t) = C\sin(\gamma t + \theta)$, then use the trig. identity $\sin(\gamma t + \theta) = \sin(\gamma t)\cos(\theta) + \cos(\gamma t)\sin(\theta)$ to show that $C = \sqrt{A^2 + B^2}$. You do not have to explicitly solve for θ.)

   (c) (5pts) The "new" form of x_p makes it obvious that x_p is an oscillation with amplitude $F_0 M(\gamma)$. In particular, it shows that the "output amplitude" depends on the "input frequency." This dependence, $M(\gamma)$, is called the "frequency response curve." Plot it for $0 \leq \gamma \leq 5$. Determine from this plot the approximate value (eyeball it) of γ_r which makes $M(\gamma)$ a maximum. Compute an more accurate value of γ using Calculus and *Mathematica* to find where the derivative is zero. This value of γ_r is called the *resonant frequency* of the spring-mass system. Why is it important?

(d) Extra Credit (3pts) Use the initial conditions for the numerical solution to solve for c_1 and c_2 in the general solution. Then compare the graph of the analytic solution with the numerical solutions obtained in part 2(a). Even better, manipulate the analytic solution and its graph like you did with the numerical solution.

Include in your conclusions a comparison of various methods of obtaining the resonant frequency and the corresponding "amplitude response."