Differential Equations with Linear Algebra, Math 3280
Lab #9: Systems of Differential Equations: Phase plane solutions

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Tues. Nov. 30, Dec. 7, 2010. Due: Wed. Dec. 8, 2010

Directions: Do the following tasks. Either provide the Lab Instructor with a writeup of your results, or have the Instructor check off each task you do. Your previous Spring-Mass Lab may be useful for problem 2. Choice of scale may be crucial, especially in problems 2 and 3.

Grading: 10 points per problem, 30 points total. Detail grading points are indicated below.

1. A linear system of two equations. Consider the system

$$\dot{x} = -y, \dot{y} = x.$$

   (3pts) Analytic solution: an analytic solution can be obtained by the "eigenvalue-eigenvector" method. Do this to find the general solution, and the specific solution corresponding to initial conditions $x(0) = 1, y(0) = 0$. (1pt) As a check, you should obtain the specific solution: $x(t) = \cos(t), y(t) = \sin(t)$. Verify either with *Mathematica* or by hand that this is in fact the solution to this IVP.

   (3pts) Numerical solution: determine a numerical solution to the same IVP using the *Mathematica* command NDSolve. Make the following plots of the numerical solution: $x(t)$, $y(t)$, and y vs x. (The y vs x plot is the *phase plane.*) For the phase plane plot, you will need to use the command ParametricPlot.

   (2pts) Graph of qualitative solution (phase plane and velocity field sketch): using the VectorPlot command (recall lab 2), plot the velocity vector field for the differential equation. Sketch several phase curves (projections of solutions in the (t, x, y) space to the (x, y) phase plane) which are consistent with the velocity vector field. You will likely use the ParametricPlot command.

   (1pt) Show that the graphs of the analytic, numerical, and graphical solutions are consistent.

2. A second order linear differential equation converted to a system of differential equations. The differential equation we investigated for the spring-mass lab was

$$25\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 50x = F_0 \cos(\gamma t) \qquad (1)$$

   When converted to a system (make sure you know how to do this), it becomes

$$\dot{x} = y, \dot{y} = -2x - .4y + \frac{F_0}{25}\cos(\gamma t)$$

   (a) (2pts) Choose parameter values of $F_0 = 0$ and $\gamma =$ anything. Numerical solutions: compute two numerical solutions using NDSolve and any two sets of initial conditions you would like to specify. Plot the two (projections of) solutions in the (x, y) phase plane. (1pt) Are these solutions consistent with what you would expect for a spring-mass system? Explain. (2pts) Sketch by hand, or use *Mathematica* to plot the vector field or direction field in the (x, y) plane. By hand, sketch several phase curves on the vector field plot/sketch. Compare with the numerically obtained phase curves.

   (b) Now consider the full nonhomogeneous differential equation with $F_0 \neq 0$.

   Analytic solution: given in spring-mass lab. Recall from Lab 4 the particular solution to equation (1) was

$$x_p(t) = \frac{F_0}{100 - 96\gamma^2 + 25\gamma^4}[(2 - \gamma^2)\cos(\gamma t) + \frac{2}{5}\gamma \sin(\gamma t)].$$

   (1pt) You plotted particular (analytic) solutions with $F_0 = 4$ and $\gamma = .5, 1.386$, and 3.0. Either replot these three solutions using Mathematica now, or bring the graphs of these three solutions from your spring-mass lab writeup.

   (2pts) Numerical solutions: use NDSolve to obtain and then plot the three full solutions (not just the particular part), one for each of the three values of γ. Compare these three numerically computed graphs to the three solutions we plotted in the spring-mass lab.

   (1pt) Why doesn't it matter (much) what initial conditions you choose? (Hint: Recall from the spring-mass lab what happens to the homogeneous part of the solution as $t \to \infty$?) (1pt) Why am I not asking you to obtain a qualitative solution using a vector field plot?

3. A nonlinear system. Assume a Rabbit and Fox population, measured in 100's, behaves according to the differential equations:

$$\dot{F} = -0.5F + \frac{FR}{.25 + R}, \quad \dot{R} = R - R^2 - \frac{RF}{.25 + R}$$

(3pts) Find all equilibria. (Hint: there are three.) (5pts) Use numerical OR graphical (velocity vector field) techniques to determine and describe the long-term bahavior of the Rabbit and Fox populations and how this behavior depends on the initial populations. (2pts) Use the Manipulate command to experiment with several initial conditions, including starting with 100 foxes and 300 rabbits. (Good luck if you try to find an analytic solution.)